Fibonacci Nos. a Multiple of 10,000

The USSR Olympiad Problem Book, by D. O. Shklarski, N . N. Chentzov, and I. M. Yaglom.

#95.* Let $\{u_n\}_{n=0}^{\infty} = 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \cdots$ be the Fibonacci numbers (indexing starts at 0, so $u_0 = 0,\ u_1 = 1,\ u_2 = 1, \cdots$). Is there a number terminating with four zeroes among the first 100,000,001 Fibonacci numbers?

Proof. Start with one zero rather than four. $u_{15} = 610$ and $u_{30} = 832,040$, and a Python program suggests that this pattern continues; namely, that $k \equiv 0 \pmod{15} \implies u_n \equiv 0 \pmod{10}$. The first step to proving this is to note that the Fibobnacci numbers follow the pattern even – odd – odd – even – odd – odd – ... That is:

$$k \equiv 0 \pmod 3 \implies u_k \equiv 0 \pmod 2,\ u_{k+1} \equiv u_{k+2} \equiv 1 \pmod 2. \tag{1}$$

The proof is by induction. For suppose (1) holds for k. It suffices to prove that (1) holds with k replaced by $k + 3$. Then:

$$\begin{aligned} u_{k+3} &\equiv u_{k+2} + u_{k+1} \equiv 1 + 1 \equiv 0 \pmod 2, \\ u_{k+4} &\equiv u_{k+3} + u_{k+2} \equiv 0 + 1 \equiv 1 \pmod 2, \\ u_{k+5} &\equiv u_{k+4} + u_{k+3} \equiv 1 + 0 \equiv 1 \pmod 2. \end{aligned}$$

These three statements prove the inductive step, so (1) holds for all k. Next note that $u_5 = 5,\ u_{10} = 55,\ u_{15} = 610$, suggesting that u_k is a multiple of 5 when k is, a hypothesis confirmed by Python for larger values of k. Again the proof is by induction. For suppose that:

$$k \equiv 0 \pmod 5 \implies u_k \equiv 0 \pmod 5. \tag{2}$$

Then:

$$\begin{aligned} u_{k+5} &= u_{k+4} + u_{k+3} \\ &= (u_{k+3} + u_{k+2}) + u_{k+3} \\ &= 2u_{k+3} + u_{k+2}) \\ &= 2(u_{k+2} + u_{k+1}) + u_{k+2} \\ &= 3u_{k+2} + 2u_{k+1} \\ &= 3(u_{k+1} + u_k) + 2u_{k+1} \\ &= 5u_{k+1} + 3u_k. \end{aligned} \tag{3}$$

This last expression is congruent to 0 mod 5 if u_k is, providing the inductive step, so (2) holds for all k. (1) and (2) together imply that:

$$k \equiv 0 \pmod{15} \implies u_k \equiv 0 \pmod{10}.$$

Note the coefficients in (3) are themselves Fibonacci numbers (5 and 3). This is not a coincidence and in fact (3) can be generalized:

Theorem 1. $u_{k+j} = u_j \cdot u_{k+1} + u_{j-1} \cdot u_k, \quad k \geq 0,\ j \geq 1.$ (4)

Proof. Use induction on j. For any k, put $j = 1$ and $j = 2$ in (4):

$$\begin{aligned} j = 1 : u_{k+1} &= u_1 \cdot u_{k+1} + u_0 \cdot u_k \\ u_{k+1} &= 1 \cdot u_{k+1} + 0 \cdot u_k \quad \checkmark \\ j = 2 : u_{k+2} &= u_2 \cdot u_{k+1} + u_1 \cdot u_k \\ u_{k+2} &= 1 \cdot u_{k+1} + 1 \cdot u_k \quad \checkmark \end{aligned}$$

Assume (4) holds for j and $j-1$ and add the two expressions:

$$\begin{aligned} u_{k+j} &= u_j \cdot u_{k+1} + u_{j-1} \cdot u_k \\ u_{k+(j-1)} &= u_{j-1} \cdot u_{k+1} + u_{(j-1)-1} \cdot u_k \\ \therefore u_{k+j} + u_{(k+j)-1} &= (u_j + u_{j-1}) \cdot u_{k+1} + (u_{j-1} + u_{j-2}) \cdot u_k \\ u_{k+(j+1)} &= u_{j+1} \cdot u_{k+1} + u_j \cdot u_k . \end{aligned}$$

The last equation is (4) with j replaced by $j+1$, proving the inductive step. **qed.**

Python suggests that: $k \equiv 0 \pmod{150} \implies u_n \equiv 0 \pmod{100}$. This can be proven in two parts, much like the case for mod 10. First put $j = 25$ in (4):

$$\begin{aligned} u_{k+25} &= u_{25} \cdot u_{k+1} + u_{24} \cdot u_k \\ u_{k+25} &= 75,025 \cdot u_{k+1} + u_{24} \cdot u_k . \end{aligned}$$

This shows that if $25 \mid u_k$, then $25 \mid u_{k+25}$; and since $u_0 = 0$:

$$k \equiv 0 \pmod{25} \implies u_k \equiv 0 \pmod{25}. \tag{5}$$

Secondly, put $j = 6$ in (4):

$$\begin{aligned} u_{k+6} &= u_6 \cdot u_{k+1} + u_5 \cdot u_k \\ u_{k+6} &= 8 \cdot u_{k+1} + u_5 \cdot u_k . \end{aligned}$$

This shows that if $4 \mid u_k$, then $4 \mid u_{k+6}$, implying that:

$$k \equiv 0 \pmod{6} \implies u_k \equiv 0 \pmod{4}. \tag{6}$$

(5) and (6) together imply that:

$$k \equiv 0 \pmod{150} \implies u_k \equiv 0 \pmod{100}. \tag{7}$$

Python suggests that $k \equiv 0 \pmod{750} \implies u_n \equiv 0 \pmod{1000}$. This is proven similarly to the last result for mod 100. Here the key is that $750 = 25 \cdot 30$. (5) provides the neeeded result for 25 and for 30 there is this:

$$u_{k+30} = u_{30} \cdot u_{k+1} + u_{29} \cdot u_k . \tag{8}$$

$u_{30} = 832,040$ is a multiple of 40, so (8) implies that if $40 \mid u_k$, then $40 \mid u_{k+30}$, and:

$$k \equiv 0 \pmod{30} \implies u_k \equiv 0 \pmod{40}. \tag{9}$$

(8) and (9) together imply that indeed:

$$k \equiv 0 \pmod{750} \implies u_k \equiv 0 \pmod{1000}. \tag{10}$$

Now it is time to tackle the problem as posed regarding Fibonacci numbers ending in four zeroes. Python suggests that $k \equiv 0 \pmod{7500} \implies u_n \equiv 0 \pmod{10,000}$; in particular, $u_{7500} \equiv 0 \pmod{10,000}$, well under the limit of 100,000,001 Fibonacci numbers specified by the problem. Take a moment to note how very large u_{7500} is at 1568 digits ($u_n \sim \varphi^n/\sqrt{5}$, where $\varphi = (\sqrt{5}+1)/2 \sim 1.618$). Here the relevant factorization is $7500 = 12 \cdot 625$. The key facts are that $16 \mid (u_{12} = 144)$ and $625 \mid u_{625}$. As earlier, write:

$$\begin{aligned} u_{k+12} &= u_{12} \cdot u_{k+1} + u_{11} \cdot u_k \\ u_{k+625} &= u_{625} \cdot u_{k+1} + u_{624} \cdot u_k. \end{aligned}$$

to see that:

$$\begin{aligned} k \equiv 0 \pmod{12} &\implies u_k \equiv 0 \pmod{16} \\ k \equiv 0 \pmod{625} &\implies u_k \equiv 0 \pmod{625}. \end{aligned}$$

These two statements imply that:

$$k \equiv 0 \pmod{7,500} \implies u_k \equiv 0 \pmod{10,000}. \tag{11}$$

WolframAlpha determines in a split second that $u_{625} \equiv 0 \pmod{625}$, but there were no computers in 1935, when these problems were first proposed. Hand calculating a 131-digit number like u_{625} at that time would've been truly grim. Carrying the calculations forward mod 625 would have considerably eased the pain, and reductions using (4) further simplify the task. Consider this reduction:

$$\begin{aligned} k = 312,\ j = 313 &: u_{625} = u_{313} \cdot u_{313} + u_{312} \cdot u_{312} \\ k = 156,\ j = 157 &: u_{313} = u_{157} \cdot u_{157} + u_{156} \cdot u_{156} \\ k = 156,\ j = 156 &: u_{312} = u_{156} \cdot u_{157} + u_{155} \cdot u_{156} \end{aligned}$$

Chaining these together, u_{625} can be expressed in terms of u_{155} and $u_{156} - u_{157}$ as well, but that too can be expressed as the sum of u_{155} and u_{156}. Replicating this kind of reduction:

- u_{625} can be expressed in terms of u_{155} and u_{156}
- u_{155} and u_{156} can be expressed in terms of u_{77} and u_{78}
- u_{77} and u_{78} can be expressed in terms of u_{38} and u_{39}
- u_{38} and u_{39} can be expressed in terms of u_{18} and u_{19}

The calculation proceeds from the bottom up:

$$\begin{aligned} u_{18} &= 2584 \equiv 84 \pmod{625} \\ u_{19} &= 4181 \equiv 431 \pmod{625} \\ u_{20} &= u_{18} + u_{19} \equiv 515 \pmod{625} \\ \therefore u_{38} &= u_{19} \cdot u_{20} + u_{18} \cdot u_{19} \equiv 431 \cdot 515 + 84 \cdot 431 \pmod{625} \\ &\equiv 44 \pmod{625} \\ u_{39} &= u_{20} \cdot u_{20} + u_{19} \cdot u_{19} \equiv 515 \cdot 515 + 431 \cdot 431 \pmod{625} \\ &\equiv 361 \pmod{625}. \end{aligned}$$

Working up to u_{625} is not particularly onerous, even with paper and pencil, and produces the result that $u_{625} \equiv 0 \pmod{625}$, as claimed.

The final result is that (11) holds, so u_{7500} ends in four zeroes, and in fact u_m ends in four zeroes whenever m is a multiple of 7500. **QED.**

The first four rows below have been proven above, the last two are shown by Python and WolframAlpha:

n	#0s at the end of u_n
15	1
150	2
750	3
7500	4
75,000	5
750,000	6

It was proven for the first four rows that not only the value of n in the chart but also every multiple of it, kn, has the indicated number of zeroes at the end of u_{kn}. This is also true for the last two rows, which is proven just like the case for $n = 7500$ immediately above.

Finally, these values of n are the *only* ones such that u_n ends in the specified number of zeroes. For one zero, this amount to the statement:

$$n \not| 15 \implies u_n \text{ does not end in zero.} \tag{12}$$

But the stronger statement

$$n \not| 3 \implies u_n \text{ is odd} \tag{13}$$

is implied by (1), proven at the top of this document. This proves (12). In fact, (13) proves that the solutions implicit in the chart above are the only solutions, considering that each value of n is a multiple of 3.

– Mike Bertrand
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