Sums of Powers of Integers

The USSR Olympiad Problem Book, by D. O. Shklarski, N . N. Chentzov, and I. M. Yaglom.

#316. (a) Prove that the sum $1^k + 2^k + 3^k + \cdots + n^k$ is a polynomial in n of degree $k + 1$.
(b) Calculate the coefficients of n^{k+1} and n^k of this polynomial.

Proof. Put $s_p(n) = \sum_{k=1}^{n} k^p = 1^p + 2^p + 3^p + \cdots + n^p$. Note $s_0(n) = n$ and $s_1(n) = n(n + 1)/2$. Proceed as follows to derive $s_2(n)$ from these two:

$$\begin{aligned} (n+1)^3 - n^3 &= 3n^2 + 3n + 1 \\ n^3 - (n-1)^3 &= 3(n-1)^2 + 3(n-1) + 1 \\ (n-1)^3 - (n-2)^3 &= 3(n-2)^2 + 3(n-2) + 1 \\ &\vdots \\ 2^3 - 1^3 &= 3 \cdot 1^2 + 3 \cdot 1 + 1 \\ 1^3 - 0^3 &= 0^2 + 3 \cdot 0 + 1 \end{aligned}$$

When adding these equations, the left side telescopes to give:

$$\begin{aligned} (n+1)^3 &= 3 \cdot \sum_{k=0}^{n} k^2 + 3 \cdot \sum_{k=0}^{n} k + n + 1 \\ &= 3 \cdot s_2(n) + 3 \cdot s_1(n) + s_0(n) + 1. \end{aligned} \tag{1}$$

All the elements in equation (1) are known except for $s_2(n)$, which can therefore be calculated from (1). The left side of (1) has degree 3, but $s_1(n)$ and $s_0(n)$ have degree 2 and 1 respectively, so $s_2(n)$ is a polynomial in n of degree 3. Let α_3 be the leading coefficient of $s_2(n)$. The leading coefficient on the left side of (1) (the coefficient of n^3) is 1 and the leading coefficient on the right side of (1) is $3\alpha_3$, so $s_2(n)$ is a third degree polynomial in n with leading coefficient $\alpha_3 = 1/3$. Let β_3 be the coefficient of second term in $s_2(n)$, that is, β_3 is the coefficient of n^2. The coefficient of n^2 on the left side of (1) is 3 and the coefficient of n^2 on the right side of (1) is $3\beta_3 + 3/2$ (the 3/2 is because the coefficient of n^2 of $s_1(n)$ is 1/2). That is:

$$\begin{aligned} 3 &= 3\beta_3 + 3/2 \\ \therefore 1 &= \beta_3 + 1/2 \\ \beta_3 &= 1/2. \end{aligned}$$

This solves the problem for $k = 2$ an makes possible the calculation of the other coefficients of $s_2(n)$. Let γ_3, δ_3 be the coefficient of n and the constant term of $s_2(n)$. (1) reduces to this taking into account only the linear and constant terms:

$$3n + 1 = 3(\gamma_3 \cdot n + \delta_3) + 3 \cdot \left(\frac{1}{2} \cdot n\right) + n + 1$$

$$\therefore \frac{1}{2} \cdot n = 3 \cdot \gamma_3 \cdot n + 3\delta_3$$

It follows that $\gamma_3 = 1/6,\ \delta_3 = 0$ and that:

$$s_2(n) = \sum_{k=0}^{n} k^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n.$$

The method employed here to calculate $s_2(n)$ from $s_1(n)$ and $s_0(n)$ can be used to calculate $s_p(n)$ from the earlier $s_q(n)$, where $0 \leq q < p$. The general formula analogous to (1) and derived in the same way, is:

$$(n+1)^{p+1} - 1 = \sum_{k=0}^{p} \binom{p+1}{k} s_k(n), \quad n \geq 0,\ p \geq 1. \tag{2}$$

The proof proceeds by complete induction. Assume that for all $1 \leq q < p$, $s_q(n)$ is a polynomial in n of degree $q+1$ with $\alpha_q = 1/(q+1)$ and $\beta_q = 1/2$. Separate out the pth summand on the right side of (2):

$$\begin{aligned}(n+1)^{p+1} - 1 &= \binom{p+1}{p} s_p(n) + \sum_{k=0}^{p-1} \binom{p+1}{k} s_k(n). \\ &= (p+1)s_p(n) + \sum_{k=0}^{p-1} \binom{p+1}{k} s_k(n). \end{aligned} \tag{3}$$

By the inductive hypothesis, the sum on the right side of (3) is a polynomial in n of degree p. The polynomial on the left side of (3) has degree $p+1$ and has leading coefficient 1. Only $s_p(n)$ can contribute to the n^{p+1} term on the right side, so the coefficient of n^{p+1} on the right side is $(p+1) \cdot \alpha_p$. It follows by induction that $\alpha_p = 1/(p+1)$ and that $s_p(n)$ has degree $p+1$.

All three pieces of (3) contribute to the coefficient of n^p:

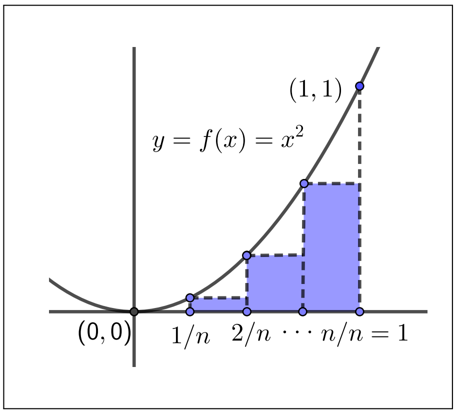
$$p + 1 = (p+1) \cdot \beta_p + \binom{p+1}{p-1} \cdot \frac{1}{p}. \tag{4}$$

Let's take these three pieces one after the other in order to justify (4). The coefficient of n^p on the left side of (3) is $p+1$ by expanding $(n+1)^{p+1}$. By definition, β_p is the coefficient of the next-to-highest power of $s_p(n)$, a polynomial of degree $p+1$ — that is, β_n is the coefficient of n^p in $s_p(n)$. By (3), β_p is multiplied by $p+1$. For the third part, the sum, the only contribution to n^p comes when $k = p-1$, which is the term appearing on the right of (4). Continuing from (4):

$$\begin{aligned} p + 1 &= (p+1) \cdot \beta_p + \frac{(p+1) \cdot p}{2} \cdot \frac{1}{p} \\ \therefore 1 &= \beta_p + \frac{1}{2} \\ \beta_p &= \frac{1}{2}. \end{aligned}$$

This establishes that for $p = 2, 3, 4, \cdots$, $s_p(n)$ is a polynomial in n of degree $p+1$ with leading coeficient $1/(p+1)$ and the next-to-leading coefficient $1/2$. **QED.**

Blaise Pascal developed this approach in the seventeenth century in *Potestatum Numericarum Summa* (The Sum of Powers of Numbers), a short appendex to his account of what is now known as Pascal's Triangle.



Pascal used these formulas to find areas under polynomial curves much as we would today. The figure here shows a Riemann sum for $f(x) = x^2$ between $x = 0$ and $x = 1$, where the partition points are equally spaced between 0 and 1.

The heights of the rectangles are given by $f(1/n)$, $f(2/n)$, $\cdots f((n-1)/n)$ and the width of each of them is $1/n$, so their areas can be easily summed. The idea is to let n increase indefinitely so the rectangles get thinner and the sum of their areas gets closer and closer to the area under the curve, defined to be $\int_0^1 x^2 dx$, the *integral* of $f(x) = x^2$ from 0 to 1. The process is called *integrating* $f(x)$ from 0 to 1. The sum of these rectangles $S_n(x^2)$ is a Riemann sum and is calculated like this:

$$\begin{aligned} S_n(x^2) &= \frac{1}{n} \cdot f\left(\frac{1}{n}\right) + \frac{1}{n} \cdot f\left(\frac{2}{n}\right) + \cdots + \frac{1}{n} \cdot f\left(\frac{n-1}{n}\right) \\ &= \frac{1}{n} \cdot \left(\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{n-1}{n}\right)^2\right) \\ &= \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + (n-1)^2\right) \\ &= \frac{1}{n^3}\left(\frac{1}{3}(n-1)^3 + \text{ polynomial in } n \text{ of degree } 2\right) \\ &= \frac{1}{n^3}\left(\frac{1}{3}n^3 + \text{ polynomial in } n \text{ of degree } 2\right) \\ &= \frac{1}{3} + \frac{\text{polynomial in } n \text{ of degree } 2}{n^3} \longrightarrow \frac{1}{3}. \end{aligned}$$

So:

$$\int_0^1 x^2 dx = \frac{1}{3},$$

and similarly since $s_p(n) = 1/(p+1) \cdot x^{p+1} +$ a polynomial in n of degree p:

$$\int_0^1 x^p dx = \frac{1}{p+1}.$$

If $g(x) = ax^p + bx^q$, where $a \geq 0$, $b \geq 0$ and p and q are integers greater than 0, then the areas add and scale:

$$\begin{aligned} \int_0^1 g(x)dx &= \int_0^1 (ax^p + bx^q)dx \\ &= \int_0^1 ax^p dx + \int_0^1 bx^q dx \\ &= a\int_0^1 x^p dx + b\int_0^1 x^q dx \\ &= \frac{a}{p+1} + \frac{b}{q+1}. \end{aligned}$$

Extending the argument to any number of summands provides a formula for integrating any polynomial with non-negative coefficients from 0 to 1.

Change the upper limit from 1 to some positive number u. Then the calculation for $S_n(x^2)$ can be replicated by scaling the diagram / calculation by u horizontally — again there are n rectangles, but now each has a width of u/n and the ordinates are $u/n,\ 2u/n,\ \cdots$. Working through he calculation, there will be a multiplier of u^3 in front of the sum, so:

$$\int_0^u x^2 dx = \frac{u^3}{3},$$

and similar considerations result in:

$$\int_0^u x^p dx = \frac{u^{p+1}}{p+1}.$$

The discussion can be continued in natural ways to adjust the lower limit and provide for negative coefficients (if the curve runs under the x-axis, the integral becomes negative!). The upshot is that Pascal's approach enables the integration of any polynomial. **QED.**

– Mike Bertrand
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