$(a + 1/a)^2 + (b + 1/b)^2 \geq 25/2$

The USSR Olympiad Problem Book, by D. O. Shklarski, N . N. Chentzov, and I. M. Yaglom.

#258. Prove that if $a + b = 1$, where a and b are positive numbers, then:

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq \frac{25}{2}.$$

Proof. First is to prove that $A^2 + B^2 \geq 2 \cdot ((A+B)/2)^2$ for real numbers A and B. Suppose it's true:

$$A^2 + B^2 \geq 2 \cdot \left(\frac{A+B}{2}\right)^2 \tag{1}$$

$$\begin{aligned} \therefore 2(A^2 + B^2) &\geq (A+B)^2 \\ 2A^2 + 2B^2 &\geq A^2 + 2AB + B^2 \\ A^2 + B^2 &\geq 2AB \\ A^2 - 2AB + B^2 &\geq 0 \\ (A - B)^2 &\geq 0. \end{aligned}$$

The last statement is true for all A, B, and these steps are reversible, so (1) holds for all A, B as well. Note that (1) is an equality when, and only when, $A = B$. Make the substitutions $A = 1 + 1/a,\ b = 1 + 1/b$ into (1), where $a + b = 1$:

$$\begin{aligned} \left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 &\geq 2 \cdot \left(\frac{(a + 1/a) + (b + 1/b)}{2}\right)^2 \\ &= \frac{1}{2} \cdot \left(\left(a + \frac{1}{a}\right) + \left((1-a) + \frac{1}{1-a}\right)\right)^2 \\ &= \frac{1}{2} \cdot \left(\frac{1}{a} + 1 + \frac{1}{1-a}\right)^2. \end{aligned} \tag{2}$$

The original inequality will be proven if the expression inside the parentheses in (2) is greater than or equal to 5 when $0 < a < 1$ — that is, if $1/a + 1/(1-a) \geq 4$ when $0 < a < 1$. Assume so and derive consequences:

$$\frac{1}{a} + \frac{1}{1-a} \geq 4, \quad 0 < a < 1 \tag{3}$$

$$\begin{aligned} (1-a) + a &\geq 4a(1-a) \\ 1 &\geq 4a - 4a^2 \\ 4a^2 - 4a + 1 &\geq 0 \\ (2a-1)^2 &\geq 0. \end{aligned}$$

Note this chain of deductions is only guaranteed to be valid when both a and $1 - a$ are positive and $a = 0$ and $a = 1$ are disallowed because a and $1 - a$ appear in denominators. That is, $0 < a < 1$ is assumed. In that case, each deduction follows, and the steps are all reversible, so (3) holds and the original inequality follows with equality only when $a = b = 1/2$. **QED.**

– Mike Bertrand
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