Sum of Distances to the Vertices of a Regular Polygon

The USSR Olympiad Problem Book, by D. O. Shklarski, N . N. Chentzov, and I. M. Yaglom.

#234. (a). On a circle which circumscribes an n-sided (regular) polygon $A_1A_2\cdots A_n$, a point M is taken. Prove that the sum of the squares of the distances from this point to all the vertices of the polygon is a number independent of the position of the point M on the circle , and that this sum is equal to $2nR^2$, where R is the radius of the circle.

Proof. Relabel the point M to z. Let Γ_n be the indicated sum and assume $R = 1$. Assume that the polygon is an equilateral triangle, so $n = 3$. Proceed as follows to calculate Γ_3. Let $\xi_0,\ \xi_1,\ \xi_2$ be the third roots of unity $= 1,\ \cos 2\pi/3 + i \sin 2\pi/3,\ \cos 2\pi/3 - i \sin 2\pi/3$. The ξ_i can be taken as the vertices of the triangle without loss of generality, since the orrientation of the polygon is immaterial to the calculation. Then:

$$\begin{aligned}\Gamma_3 &= |z-1|^2 + |z-\xi_1|^2 + |z-\xi_2|^2 \\ &= (z-1)(\overline{z}-1) \\ &\quad + (z-\xi_1)(\overline{z}-\overline{\xi_1}) \\ &\quad + (z-\xi_2)(\overline{z}-\overline{\xi_2}) \\ &= \boldsymbol{z\overline{z}} - \overline{z} - z + \mathbf{1} \\ &\quad + \boldsymbol{z\overline{z}} - \xi_1\overline{z} - \overline{\xi_1}z + \boldsymbol{\xi_1\overline{\xi_1}} \\ &\quad + \boldsymbol{z\overline{z}} - \xi_2\overline{z} - \overline{\xi_2}z + \boldsymbol{\xi_2\overline{\xi_2}}.\end{aligned} \tag{1}$$

Because $z,\ \xi_1$, and ξ_2 are on the unit circle, each of the six bolded terms equal 1. Therefore:

$$\begin{aligned}\Gamma_3 &= 6 - \overline{z}(1+\xi_1+\xi_2) - z(1+\xi_1+\xi_2) \\ &= 6.\end{aligned}$$

This is because $1+\xi_1+\xi_2 = 0$, easy to see in this case, but look at it this way. The third roots of unity are the roots of $w^3 - 1 = 0$ in the complex plane, arrayed around the circle. But:

$$w^3 - 1 = (w-1)(w-\xi_1)(w-\xi_2). \tag{2}$$

If the right side of (2) is multiplied out, the coefficient of w^2 is $-(1+\xi_1+\xi_2)$. But the coefficient of w^2 on the left side of (2) is zero, so $-(1+\xi_1+\xi_2) = 0$. This argument obtains for the nth roots of unity for any $n = 2, 3, 4, \ldots$, whose sum is always zero. The calculation for Γ_n is the same in all essentials, the difference being that in (1), there are n terms instead of three, leading to:

$$\begin{aligned}\Gamma_n &= 2n - (\overline{z}+z)\cdot(\text{ sum of the } n\text{th roots of unity}) \\ &= 2n - (\overline{z}+z)\cdot 0 \\ &= 2n.\end{aligned} \tag{3}$$

If the radius of the circle is R instead of 1, all the terms on the right side of (1) have a multiplier of R^2, which carries over to (3), so $\Gamma_n = 2nR^2$. **QED.**

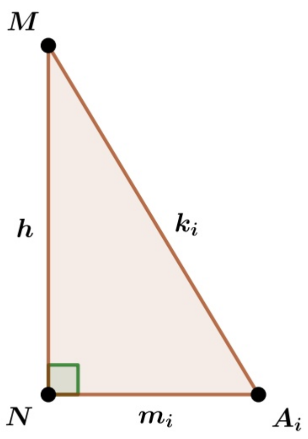
(b) Prove that the sum of the squares of the distances from an arbitrary point M, taken in the plane of a regular n-sided polygon $A_1 A_2 \cdots A_n$ to all the vertices of the polygon, depends only on the distance l of the point M from the center O of the polygon, and is equal to $n(R^2 + l^2)$, where R is the radius of the circle circumscribing the regular n-sided polygon.

Proof. Relabel the point M to z. Let $\Gamma_n(l)$ be the indicated sum and assume $R = 1$. The calculation for $\Gamma_3(l)$ is the same as in (1) and following, except that now $|z| = l$ is not necessarily 1. Collecting terms at the last step:

$$\begin{aligned}\Gamma_3(l) &= 3z \cdot \overline{z} + (1 + \xi_1 \cdot \overline{\xi_1} + \xi_2 \cdot \overline{\xi_2}) \\ &= 3 \cdot |z|^2 + 3 \\ &= 3 \cdot (l^2 + 1).\end{aligned}$$

Promoting 3 to n and 1 to R changes none of the essentials in the calculation, resulting in $\Gamma_n(l) = n \cdot (l^2 + R^2)$. **QED.**

(c) Prove that statement (b) remains correct even when point M does not lie in the plane of the n-sided polygon $A_1 A_2 \cdots A_n$.



Proof. Let N the point in the the plane of the polygon directly beneath M, so N plays the role of M in part (b). Let h be the perpendicular distance from M to the plane — that is, from M to N. If m_i is the distance from N to A_i, then part (b) provides a formula for $\sum m_i^2$. Connect M and each A_i as suggested in the diagram and let k_i be the distance from M to A_i. $k_i^2 = m_i^2 + h^2$, so:

$$\begin{aligned}\sum_{i=1}^{n} k_i^2 &= \sum_{i=1}^{n} (m_i^2 + h^2) \\ &= \sum_{i=1}^{n} m_i^2 + nh^2 \\ &= n \cdot (l^2 + R^2) + nh^2 \\ &= n \cdot (l^2 + (R^2 + h^2)) \\ &= n \cdot (l^2 + D^2),\end{aligned}$$

where $D^2 = R^2 + h^2$, so D is the distance from M to the center of the polygon in the plane. **QED.**

– Mike Bertrand
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