$(1 - x + x^2 + ...)(1 + x + x^2 + ...)$

The USSR Olympiad Problem Book, by D. O. Shklarski, N . N. Chentzov, and I. M. Yaglom.

#198. Prove that in the product

$$\left(1 - x + x^2 - x^3 + \cdots - x^{99} + x^{100}\right)\left(1 + x + x^2 + x^3 + \cdots + x^{99} + x^{100}\right),$$

after multiplying and collecting terms, there does not appear a term in x of odd degree.

Proof. Put:

$$A_n = 1 - x + x^2 - x^3 + \cdots + (-1)^n \cdot x^n$$
$$B_n = 1 + x + x^2 + x^3 + \cdots + x^n,$$

so the product in question is $A_{100} \cdot B_{100}$. Note that:

$$A_n + B_n = 2 + 2x^2 + 2x^4 + \cdots, \text{ where all the exponents are even}$$
$$A_n - B_n = -2x - 2x^3 - 2x^5 + \cdots, \text{ where all the exponents are odd.}$$

Here are the first few products:

$$A_1 \cdot B_1 = 1 - x^2$$
$$A_2 \cdot B_2 = 1 + x^2 + x^4$$
$$A_3 \cdot B_3 = 1 + x^2 - x^4 - x^6$$
$$A_4 \cdot B_4 = 1 + x^2 + x^4 + x^6 + x^8.$$

The proof is by induction. Assume that $A_n \cdot B_n$ contains only even powers when multiplied out. Then

$$\begin{aligned} A_{n+1} \cdot B_{n+1} &= (A_n + (-1)^n x^n) \cdot (B_n + x^n) \\ &= A_n B_n + (A_n + (-1)^n B_n)\, x^n + (-1)^n \cdot x^{2n} \qquad (1) \end{aligned}$$

The first term contains only even powers by inductive assumption and the third term $\pm x^{2n}$ is an even power, so the focus is on the red term in the middle. The analysis varies depending on whether n is even or odd:

$$n \text{ even} \implies (A_n + (-1)^n B_n)\, x^n = (A_n + B_n) x^n = \text{polynomial with only even powers.}$$

This is because $A_n + B_n$ contains only even powers and and x^n is an even power, so the same is true of their product. Also:

$$n \text{ odd} \implies (A_n + (-1)^n B_n)\, x^n = (A_n - B_n) x^n = \text{polynomial with only even powers.}$$

In this case, $A_n - B_n$ contains only odd powers and x^n is an odd power, so the product's exponents are all of the form odd + odd and are therefore even. In either case, the red term of (1) contains only even powers, so the same is true of $A_{n+1} \cdot B_{n+1}$. **QED.**

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