Initial and Final Digit

The USSR Olympiad Problem Book, by D. O. Shklarski, N . N. Chentzov, and I. M. Yaglom.

#19 (a). Find the smallest integer whose first digit is 1 and which has the property that if this digit is transferred to the end of the number the number is tripled.

Proof. Suppose $x = 1a$ is a two-digit solution, so:

$$\begin{aligned} 10a + 1 &= 3(10 + a) \\ 10a + 1 &= 30 + 3a \\ \therefore 7a &= 29. \end{aligned}$$

This is impossible because 7 does not divide 29. Try $x = 1ab$, leading to:

$$\begin{aligned} 100a + 10b + 1 &= 3(100 + 10a + b) \\ 100a + 10b + 1 &= 300 + 30a + 3b \\ \therefore 70a + 7b &= 299. \end{aligned} \tag{1}$$

This too is impossible because 7 does not divide 299 though it does divide the left side of equation (1). They all go like this until $7 \mid 2999\ldots999$ and that happens eventually. To see this, let $x_n = 2999\ldots999$, where there are n 9s. There is a recurrence: $x_1 = 29,\ x_n = 10x_{n-1}+9$ for $n > 1$. The issue is to find n such that $x_n \equiv 0 \pmod 7$. Look at the recurrence mod 7: $x_1 \equiv 1 \pmod 7,\ x_{n+1} \equiv 10x_n + 9 \equiv 3x_n + 2 \pmod 7$:

$$\begin{aligned} x_1 &\equiv 1 \pmod 7 \\ x_2 &\equiv (3x_1 + 2) \pmod 7 \equiv 5 \pmod 7 \\ x_3 &\equiv (3x_2 + 2) \pmod 7 \equiv 3 \pmod 7 \\ x_4 &\equiv (3x_3 + 2) \pmod 7 \equiv 4 \pmod 7 \\ x_5 &\equiv (3x_4 + 2) \pmod 7 \equiv 0 \pmod 7. \end{aligned}$$

That is, $x_5 = 299,999$ is a multiple of 7 $(299,999 = 7 \cdot 42,857)$ and:

$$\begin{aligned} 70,000a + 7000b + 700c + 70d + 7e &= 299,999 \\ \therefore 10,000a + 1000b + 100c + 10d + e &= 42,857. \end{aligned} \tag{2}$$

In equation (2), the left side is equal to e mod 10 and the right side is equal to 7 mod 10, and since $0 \le e \le 9,\ e = 7$. (2) can be reduced by 7, then divided by 10, to show that $d = 5$ and so on for the rest of the digits. It follows that the first solution is $x = 1abcde = 142,857$, where $3x = abcde1 = 428,571$. **QED.**

(b) With what digits is it possible to begin a (nonzero) integer such that the integer will be tripled upon the transfer of the initial digit to the end? Find all such integers.

Proof. Continue the discussion above to find all solutions for the digit 1. Advancing the modular calculations for x_n:

$$x_6 \equiv (3x_5 + 2) \pmod 7 \equiv 2 \pmod 7$$
$$x_7 \equiv (3x_6 + 2) \pmod 7 \equiv 1 \pmod 7.$$

That is, $x_7 \equiv x_1 \equiv 1 \pmod 7$ and the values cycle with period 6 from that point. In particular $x_5 \equiv x_{11} \equiv x_{17} \equiv \ldots \equiv 0 \pmod 7$. As in the discussion around the first solution, the only possible further solutions are connected with x_{11}, x_{17}, and so on. And each of these *does* provide a solution. Consider:

$$\frac{x_5 + 1}{7} = \frac{299999 + 1}{7} = \frac{300000}{7} = 42857.\underbrace{142857\ 142857\ 142857\ldots}_{1/7} \tag{3}$$

Note that subtracting 1/7 from equation (3) (fraction on the left, decimal on the right) leads to $x_5/7 = 42,857$ as derived above. Multiplying (3) by 10^6 results in:

$$\frac{x_{11} + 1}{7} = \frac{3 \cdot 10^{11}}{7} = 42857\ 142857.\underbrace{142857\ 142857\ldots}_{1/7} \tag{4}$$

Subtracting 1/7 from (4) results in $u_{11} = x_{11}/7 = 42857\ 142857$. It remains to show that putting a leading digit 1 in front of u_{11} results in a value one third of that arising from tacking on a trailing digit 1. Put:

$$m = u_{11} \text{ after adding a leading digit 1 } = 1 \cdot 10^{11} + u_{11}$$
$$n = u_{11} \text{ after adding a trailing digit 1 } = 10 \cdot u_{11} + 1.$$

$3 \cdot m = n$ is what's wanted, so assume that and work backwards:

$$3 \cdot m = n$$
$$3 \cdot (10^{11} + u_{11}) = 10 \cdot u_{11} + 1$$
$$3 \cdot 10^{11} + 3u_{11} = 10u_{11} + 1$$
$$3 \cdot 10^{11} - 1 = 7u_{11}. \tag{5}$$

Equation (5) is the definition of $u_{11} = x_{11}/7$, so it is true; and since all these steps are reversible, $3 \cdot m = n$ as desired and 142857 142857 is a solution. Similar calculations show how each of x_{17}, x_{24} and so on leads to a solution. Here are the first few and the pattern is clear:

$$x_5 \longrightarrow 142857$$
$$x_{11} \longrightarrow 142857\ 142857$$
$$x_{17} \longrightarrow 142857\ 142857\ 142857$$
$$x_{24} \longrightarrow 142857\ 142857\ 142857\ 142857.$$

Using a similar method for the digit 2 results in $y_1 = 58$ $y_2 = 598$ $y_3 = 5998$, and generally $y_n = 5999\ldots9998$, with $n - 1$ 9s, leading to the recurrence $y_1 = 58,\ y_n = 10y_{n-1} + 18$ for $n > 1$. As with the digit 1, $y_n \equiv 0 \pmod 7$ is necessary for a solution and the residues cycle with period 6, hitting zero the first time for $y_5 = 599998$. Divide by 7 to get the key number to append the digits 2 to, front and back: $v_5 = y_5/7 = 85714$ and this generates a solution:

$$3 * 285714 = 857142.$$

As for the digit 1, the only possible solutions remaining arise from y_{11}, y_{17}, and so on and these are indeed solutions:

$$
\begin{aligned}
y_5 &\longrightarrow 285714 \\
y_{11} &\longrightarrow 285714\ 285714 \\
y_{17} &\longrightarrow 285714\ 285714\ 285714 \\
y_{24} &\longrightarrow 285714\ 285714\ 285714\ 285714.
\end{aligned}
$$

Next is the digit 3. Let $x = a_0 \cdot 10^n + a_1 \cdot 10^{n-1} + \ldots + a_{n-1} \cdot 10 + a_n$, where:

$$3 \cdot (3a_0a_1 \ldots a_{n-1}a_n) = a_0a_1 \ldots a_{n-1}a_n3, \tag{6}$$

the 3s (apart from the one at the very start) and the a_i being digits in the decimal representations of the two numbers. Expanding equation (6):

$$
\begin{aligned}
9 \cdot 10^{n+1} + 3 \cdot a_0 \cdot 10^n + \ldots &= a_0 \cdot 10^{n+1} + \ldots + 3. \\
\therefore 9 \cdot 10^{n+1} - 3 &= a_0 \cdot (10^{n+1} - 3 \cdot 10^n) + K \\
&= a_0 \cdot (10 \cdot 10^n - 3 \cdot 10^n) + K \\
9 \cdot 10^{n+1} - 3 &= 7 \cdot a_0 \cdot 10^n + K, \qquad (7)
\end{aligned}
$$

where K is a sum of monomials $7 \cdot a_j \cdot 10^{n-j}$, the highest power of 10 being $n-1$. The leading digit of the integer on the left side of equation (7) is 8, while the leading digit on the right is 6 or less, so equation (6) is impossible.

Similar calculations obtain for the digits 4 through 9, the discrepancy at the last step being even more pronounced, so none of these digits can be used to produce the effect asked for in the problem and the only solutions are those given above for the digits 1 and 2. **QED.**

– Mike Bertrand
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