Is $99^n + 100^n > 101^n$?

The USSR Olympiad Problem Book, by D. O. Shklarski, N . N. Chentzov, and I. M. Yaglom.

#143. Which is larger, $99^n + 100^n$ or 101^n (where n is a natural number)?

Proof. Consider the function

$$f(x) = \frac{99^x + 100^x}{101^x} = \left(\frac{99}{101}\right)^x + \left(\frac{100}{101}\right)^x. \tag{1}$$

Each term on the right side of (1), considered as a function of the real variable x, is monotonically decreasing for $x \geq 1$, so $f(x)$ too is a monotonically decreasing function of x for $x \geq 1$. Furthermore $f(x) \to 0$, and since $f(1) = 1.9703$, there will be an $x_0 > 1$ such that $f(x_0) = 1$. $f(x)$ is greater than 1 for $1 \leq x < x_0$ and $f(x)$ is less than 1 for $x > x_0$. A two-line Python program evaluating $f(n)$ for integers n between 1 and 100 produces an approximation of x_0:

n	$f(n)$
47	1.0170
48	1.0031
49	0.9894
50	0.9759

Interpolating this table, $x_0 \approx 48.2263$, so the answer to the problem is that $99^n + 100^n > 101^n$ exactly for positive integers n less than 49. Even simpler would be to ask WolframAlpha to solve $f(x) = 1$, answer $= 48.2275$. All of which is grossly ahistorical if the intent is to recapitulate the situation in the 1930s, when these problems were first posed.

Another approach well within reach at that time would be to find a values u and v such that:

$$\left(\frac{99}{101}\right)^u = \frac{1}{3}, \qquad \left(\frac{100}{101}\right)^v = \frac{2}{3}. \tag{2}$$

The thinking behind the imbalanced 1/3 and 2/3 is that the first fraction decreases about twice as fast as the second one, so maybe the the u and v satisfying these two equations would be about the same. 1/2 could be used in each, but that leads to a value of u about half that of v ($u = 34.66, v = 69.66$) and we're hoping to find the median exponent to apply to each fraction. The solutions to (2) are $u = 54.93$, $v = 40.75$, whose averge is 47.84. Evaluating $f(47), f(48), f(49)$ using log tables quickly produces the answer.

Even the little bit of function theory used here would have been considered part of advanced mathematics not properly part of the toolkit of the high school students who are supposed to solve these problems. In that vein, think of f as being a function of the discrete variable $n = 1,\ 2,\ 3\ldots$. It is still true that $f(n)$ monotonically decreases as a function of n, so there is some positive integer n_0 such that:

$$\begin{aligned} 99^n + 100^n &> 101^n, \quad \text{for } n \leq n_0 \\ 99^n + 100^n &\leq 101^n, \quad \text{for } n > n_0. \end{aligned}$$

That is:

$$\begin{aligned} 101^n - 99^n &< 100^n, \quad &\text{for } n \le n_0 \\ 101^n - 99^n &\ge 100^n, \quad &\text{for } n > n_0. \end{aligned}$$

Put $K(n) = 101^n - 99^n = (100+1)^n - (100-1)^n$ and expand each exponentiated term to get this expression for $K(n)$:

$$100^n + \binom{n}{1} \cdot 100^{n-1} + \binom{n}{2} \cdot 100^{n-2} + \binom{n}{3} \cdot 100^{n-3} + \binom{n}{4} \cdot 100^{n-4} + \cdots$$

$$-\left(100^n - \binom{n}{1} \cdot 100^{n-1} + \binom{n}{2} \cdot 100^{n-2} - \binom{n}{3} \cdot 100^{n-3} + \binom{n}{4} \cdot 100^{n-4} + \cdots\right)$$

The first two terms cancel, the next two double, and so on:

$$K(n) = 2 \cdot \left(\binom{n}{1} \cdot 100^{n-1} + \binom{n}{3} \cdot 100^{n-3} + \binom{n}{5} \cdot 100^{n-5} + \cdots\right) \tag{3}$$

The boundary $n = n_0$ is such that $K(n) \approx 100^n$ and considering that $\binom{n}{1} = n$ in (3) and the other terms are relatively negligible, $n_0 = 50$ makes $K(50) = 100^{50}$ not too far off. So use (3) to evaluate $K(50)$:

$$\begin{aligned} K(50) &= 2 \cdot 50 \cdot 100^{49} + P \\ &= 100^{50} + P, \end{aligned}$$

where P is the sum of all the remaining terms, a positive value. That is, $K(50) \ge 100^{50}$, so $n_0 < 50$. Next evaluate $K(49)$:

$$\begin{aligned} K(49) &= 2 \cdot 49 \cdot 100^{48} + 2 \cdot \frac{49 \cdot 48 \cdot 47}{3!} \cdot 100^{46} + P' \\ &= 98 \cdot 100^{48} + 36,848 \cdot 100^{46} + P' \\ &= 98 \cdot 100^{48} + 3.6848 \cdot 100^{48} + P' \\ &> 100 \cdot 100^{48} + P' \\ &= 100^{49} + P', \end{aligned}$$

where P' is positive. Therefore $K(49) > 100^{49}$ and so $n_0 < 49$. Next is $K(48)$ and the object now is to show that $K(48) < 100^{48}$, so that $n_0 = 48$ as shown above:

$$\begin{aligned} \frac{K(48)}{100^{48}} &= \frac{2 \cdot 48 \cdot 100^{47}}{100^{48}} + \frac{2 \cdot 48 \cdot 47 \cdot 46 \cdot 100^{45}}{3! \cdot 100^{48}} + S \\ &= 0.96 + 0.034592 + S \\ &= 0.994592 + S, \end{aligned} \tag{4}$$

where:

$$\begin{aligned} S &= 2 \cdot \left(\frac{48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 \cdot 10^{43}}{5! \cdot 100^{48}} + \frac{48 \cdot 47 \cdot 46 \cdot 45 \cdot 44 \cdot 43 \cdot 42 \cdot 10^{41}}{7! \cdot 100^{48}} + \cdots\right) \\ &< 2 \cdot \left(\frac{48^5 \cdot 100^{43}}{5! \cdot 100^{48}} + \frac{48^7 \cdot 100^{41}}{5! \cdot 100^{48}} + \cdots\right) \\ &< 2 \cdot \left(\frac{48^5}{100^6} + \frac{48^7}{100^8} + \cdots\right) \\ &= 2 \cdot \frac{48^5/100^6}{1 - 48^2/100^2} = 0.000662. \end{aligned}$$

The fourth line here follows by summing the infinite series, which exceeds the finite number of terms in S. Therefore $S = 0.995264$ and (4) implies that $K(48) < 100^{48}$; that is, $101^{48} - 99^{48} < 100^{48}$, aligning with the previous result that the boundary is $n_0 = 48$, namely:

$$\begin{aligned} 101^n - 99^n &< 100^n, \quad \text{for } n \leq 48 \\ 101^n - 99^n &\geq 100^n, \quad \text{for } n > 48. \end{aligned}$$

QED.

– Mike Bertrand
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