Solve $x^2 + y^2 + z^2 = 2xyz$

The USSR Olympiad Problem Book, by D. O. Shklarski, N . N. Chentzov, and I. M. Yaglom.

#117. (a) Prove that the only solution in integers of the equation

$$x^2 + y^2 + z^2 = 2xyz \tag{1}$$

is $x = y = z = 0$.

Proof. The left side of equation (1) is non-negative, so the right side must be non-negative as well. Conceivably two of x, y, z in some solution are negative, but then *their* negatives plus the third non-negative value would be a solution too. The upshot is that each of x, y, z can be considered to be non-negative in any solution. If any one of x, y, z is 0, then the right side of (1) is 0 and therefore all three of x, y, z must be 0. The task is to determine if there are positive values x, y, z satisfying (1).

So suppose x, y, and z are positive and consider their parities. If exactly one of the three is odd, then the left side of (1) is odd, which is not the case. Similarly if all three are odd. Consider the posssibility that exactly two of them are odd. Because of the symmetry, it suffices to consider x to be even and y and z odd. The square of an even number is a multiple of 4 and the square of an odd number is congruent to 1 mod 4 beecause $3^2 = 9 \equiv 1 \pmod 4$. So:

$$\begin{aligned} x^2 &\equiv 0 \pmod 4 \\ y^2 &\equiv 1 \pmod 4 \\ z^2 &\equiv 1 \pmod 4 \\ \therefore x^2 + y^2 + z^2 &\equiv 2 \pmod 4. \end{aligned}$$

But $2xyz \equiv 2 \cdot 2xy \equiv 0 \pmod 4$, so this too can't be and the only possibility remaining is that all three of $x,\ y,\ z$ are even. Say $x = 2a,\ y = 2b,\ z = 2c$. Plugging into (1):

$$\begin{aligned} (2a)^2 + (2b)^2 + (2c)^2 &= 2 \cdot (2a) \cdot (2b) \cdot (2c) \\ \therefore 4a^2 + 4b^2 + 4c^2 &= 16abc \\ a^2 + b^2 + c^2 &= 4abc. \end{aligned} \tag{2}$$

The analysis applied to (1) can be applied equally well to (2), forcing all of $a,\ b,\ c$ to be even. This can be continued indefinitely, with the variable on the left being halved at each step. But such an infinite descent is impossible starting from some fixed x. It follows that equation (1) is impossible in integers except for $x = y = z = 0$. **QED.**

(b) Find the integers x, y, z, v such that

$$x^2 + y^2 + z^2 + v^2 = 2xyzv. \tag{3}$$

Proof. As in part (a), if there is any solution other than $x = y = z = v = 0$, then there is one such that all four variables are positive. Assume that x, y, z, v is such a solution. If exactly one of the four is odd, then the left side of (3) is odd, the right side even, so that can't happen. Similarly if exactly three of the four are odd. If exactly two are odd, then the left side of (3) is congruent to 2 mod 4 and the right side is congruent to 0 mod 4, so that can't be the case either. If all four are odd, then the left side of (3) is congruent to 0 mod 4, but:

$$\begin{aligned} xyzw &\equiv 1 \text{ or } 3 \pmod 4 \\ \therefore 2xyzv &\equiv 2 \pmod 4. \end{aligned}$$

This inconsistency rules out that all four variables are odd. All that remains is that all four are even. Put $x = 2a,\ y = 2b,\ z = 2c,\ v = 2d$ and plug into (2):

$$\begin{aligned} (2a)^2 + (2b)^2 + (2c)^2 + (2d)^2 &= 2 \cdot (2a) \cdot (2b) \cdot (2c) \cdot (2d) \\ \therefore 4a^2 + 4b^2 + 4c^2 + 4d^2 &= 32abcd \\ a^2 + b^2 + c^2 + d^2 &= 8abcd. \end{aligned} \tag{4}$$

The reduction of (4) is similar to that of (3) for ruling out that one, two, or three of $a,\ b,\ c,\ d$ are odd. To address the possibility that all four are odd, consider that the square of an odd number is always congruent to 1 mod 8. So in this case, the left side of (4) is congruent to 4 mod 8 and the right side is congruent to 0 mod 8, ruling out that all four are odd. All that's left is that all four of $a,\ b,\ c,\ d$ are even. Like with part (a), the process can be repeated indefinitely, which is not possible. It follows that the only solution to the original problem is $x = y = z = v = 0$. **QED.**

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