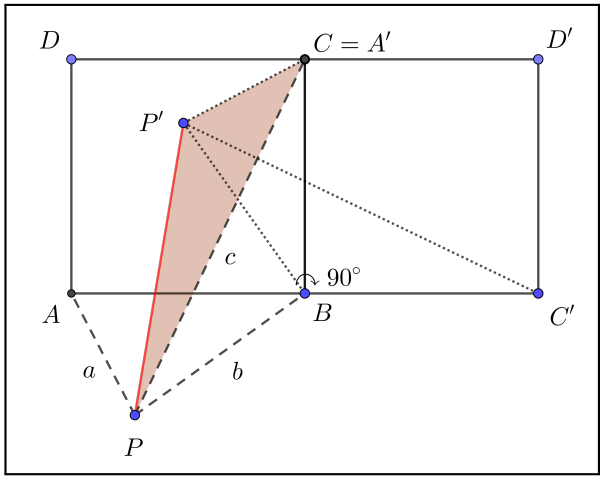
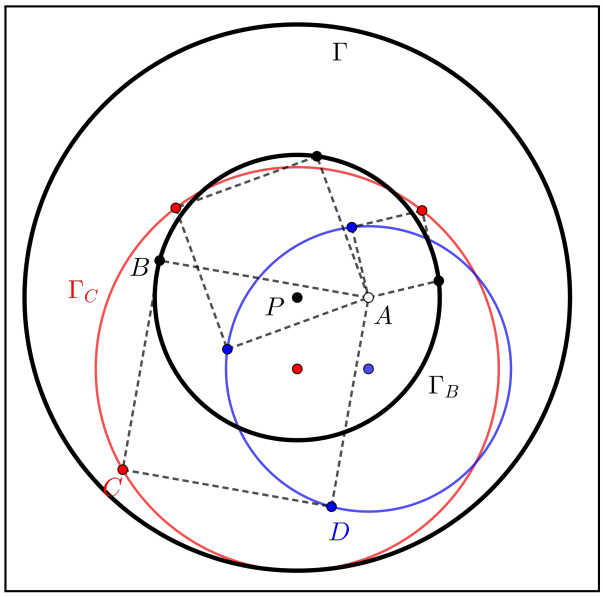
Square with Vertices at Three Distances from Point

#7. Given three distances a, b, c, construct (using straightedge and compass and without analytic geometry) a square $ABCD$ and a point P such that $|PA| = a$, $|PB| = b$, and $|PC| = c$ (*Mathematics Magazine* Problem 1418, April 1993).

Construction. This isn't possible in general. Suppose WLOG that $a \le b \le c$. Let $\square ABCD$ and point P be as pictured, where the segments PA, PB, PC are sketched with dashed lines. A is the lower-left corner of the square and the vertices are traversed counterclockwise. Rotate the square around B by 90°, so A maps to $A' = C$, D maps to D', C maps to C', and B remains fixed. P maps to P' and segments PA, PB, PC map to segments $P'C, P'B, P'C'$ respectively, drawn as dotted lines. Note that $\triangle PBP'$ is right isosceles since $P'B$ results from rotating PB by 90°. Therefore $PP'^2 = 2PB^2 = 2b^2$, so $PP' = \sqrt{2}b$. The triangle law applied to $\triangle PP'C$ implies that $c \le a + \sqrt{2}b$. This is a necessary condition for a solution, and there always *is* a solution in this case.



This image is busy, so try to focus on each element as it's being described. Point P is located in center of the diagram, denoted by a black dot. Assume $a \le b \le c$, relabeling if necessary. Draw circle Γ_B centered at P with radius b — the inner thick black circle in the diagram. Fix A as a point a units from P to the right, within and possibly on Γ_B (A is the hollow point to the right of P in the diagram). For each point B on Γ_B, construct square $ABCD$, proceeding counter clockwise in moving from A to B and then the other vertices. Compare with the diagram, where three such squares are constructed and shown in dashed lines, each moving counter-clockwise from hollow dot A to black dot B to red dot C to blue dot D. As B moves around Γ_B, the associated point C, diagonal to A on one of the squares, itself traces out a circle, Γ_C, shown in red.



The reason the Cs trace out a circle is that the transformation taking B to C is a dilative rotation about A — starting from a given point B, C is determined by rotating B by $+\pi/4$ around A, then expanding by a factor of $\sqrt{2}$. Such a transformation takes circles into circles, and the new circle Γ_C has a radius of $\sqrt{2}b$, since Γ_C is $\sqrt{2}$ times larger than Γ_B, a circle of radius b. The same transformation produces the center of Γ_C: rotating P around A by $+\pi/4$ and expanding by $\sqrt{2}$ brings us to the point a units directly below P as the center of Γ_C, the red dot below P in the diagram.

Similarly all the points D lie on another circle Γ_D, since the transformation taking B to D is also a rotation applied to the original circle Γ_B, this time by $+\pi/2$ and with no dilation. Therefore Γ_D has radius b and its center is a units directly below A (point P rotated by $+\pi/2$ around A). This is the blue circle.

By the nature of the construction, $|PA| = a$ and $|PB| = b$ for each of the squares, and all squares meeting these conditions are produced by this construction. We have a solution to the problem if and only if a point D on one of the squares is c units from point P.

Let Γ be the circle centered at P with radius $a + \sqrt{2}b$ – this is the smallest circle centered at P containing Γ_C and the encompassing thick black circle. Γ completely contains Γ_D since the point on Γ_D farthest from P is $\sqrt{2}a + b \leq a + \sqrt{2}b$. Every circle concentric to Γ_B and Γ and between them intersects Γ_C in at least one point (two except at the extremity on Γ), leading to squares satisfying the conditions of the problem. If C is a point of intersection of Γ_C and the newly drawn intermediate circle, then build out the square $ABCD$, proceeding counterclockwise from A to B and the rest. Then B is on Γ_B and D is on the blue circle. **QEF**.

– Mike Bertrand
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