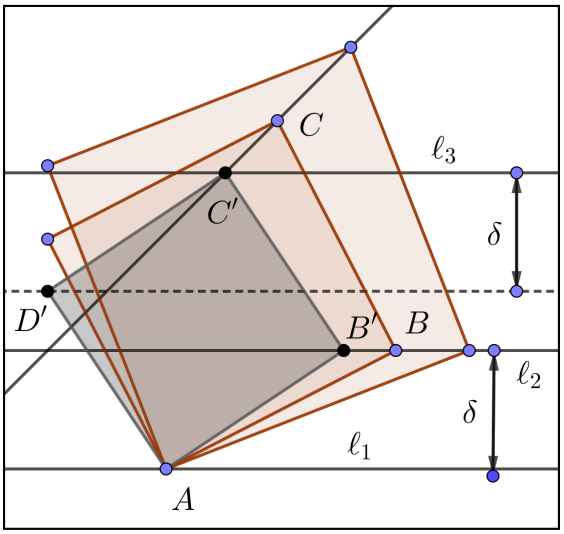
Square on Four Parallel Lines

#5. Given four distinct parallel lines, construct a square with a vertex on each line.



Construction. Let the lines be horizontal with ℓ_1 the lowest and ℓ_3 the highest. Fix a point A on ℓ_1 and for each point B on ℓ_2, construct the square on side AB riding above ℓ_1, where point C is diagonal to A. The transformation taking B to C is a rotation by $\pm\pi/4$ around A — plus if B is to the right of A, negative otherwise (counter-clockwise if plus, clockwise if negative). The diagram shows the solution for Bs taken to the right. There is another solution when the Bs are to the left, that square being the mirror image of the gray square across the vertical line through A. To find that solution, consider that rotations take lines to lines, so the image of ℓ_1 is also a line, the diagonal line through C in the diagram. Suppose this line intersects ℓ_3 at point C'. Then the square with diagonal AC' (the gray one in the diagram, $\square AB'C'D'$) has a vertex on each of ℓ_1, ℓ_2, and ℓ_3. But then *the fourth vertex of the square is completely determined* — it is point D' to the left on the gray square.

The dashed horizontal line through D' is the only possiblility for the fourth line. Since AB' and $D'C'$ are parallel and have the same length, the vertical distance δ between ℓ_3 and the dashed line is the same as the vertical distance between ℓ_2 and ℓ_1, providing the necessary and sufficient condition for there to be a solution. **QEF**.

– Mike Bertrand
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