$$\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}, \qquad \cot\frac{3\pi}{4} = -1,$$

$$\cos\frac{3\pi}{4} = \frac{-1}{\sqrt{2}}, \qquad \sec\frac{3\pi}{4} = -\sqrt{2},$$

$$\tan\frac{3\pi}{4} = -1, \qquad \csc\frac{3\pi}{4} = \sqrt{2}.$$

The values of the circular functions of any other odd multiple of $\pi/4$ can be obtained by similar methods.

Problems

Draw a figure showing each real number θ in the unit circle and verify the following by finding the exact values.

1. (a) $\sin\pi/3 = \cos\pi/6$
   (b) $\sin\pi/6 = \cos\pi/3$
2. (a) $\sin\pi/6 = \sin 5\pi/6$
   (b) $\cos 5\pi/6 = -\cos\pi/6$
3. (a) $\tan\pi/4 = \tan 5\pi/4$
   (b) $\cot 3\pi/4 = \cot 7\pi/4$
4. (a) $\sec 11\pi/6 = \sec\pi/6$
   (b) $\csc 2\pi/3 = -\csc 4\pi/3$
5. (a) $\sin 2\pi/3 = \sin(4\pi/3)$
   (b) $\cos 7\pi/6 = \cos(-5\pi/6)$
6. (a) $\sin\pi/3 = 2\sin\pi/6\cos\pi/6$
   (b) $\sin\pi/2 = 2\sin\pi/4\cos\pi/4$
7. (a) $\sin\dfrac{\pi}{6} = \sqrt{\dfrac{1-\cos\pi/3}{2}}$
   (b) $\cos\dfrac{\pi}{6} = \sqrt{\dfrac{1+\cos\pi/3}{2}}$
8. $\tan\dfrac{\pi}{6} = \dfrac{1-\cos\pi/3}{\sin\pi/3} = \dfrac{\sin\pi/3}{1+\cos\pi/3}$
9. $\cos\dfrac{\pi}{3} = \cos^2\dfrac{\pi}{6} - \sin^2\dfrac{\pi}{6} = 2\cos^2\dfrac{\pi}{6} - 1$

Find the exact numerical values of the following.

10. (a) $\sin^2\dfrac{\pi}{6} + \cos^2\dfrac{\pi}{6}$ (b) $\sin^2 0 + \cos^2 0$
11. (a) $\sec^2\dfrac{\pi}{3} - \tan^2\dfrac{\pi}{3}$ (b) $\sec^2\dfrac{5\pi}{4} - \tan^2\dfrac{5\pi}{4}$
12. (a) $\csc^2\dfrac{7\pi}{4} - \cot^2\dfrac{7\pi}{4}$ (b) $\csc^2\dfrac{3\pi}{4} - \cot^2\dfrac{3\pi}{4}$
13. $\sin\dfrac{2\pi}{3} + \cos\dfrac{7\pi}{6} + \tan\dfrac{5\pi}{3}$

14. $\tan\dfrac{5\pi}{4} + \cot\dfrac{7\pi}{4} - \sec\dfrac{5\pi}{6}$
15. $\csc\dfrac{5\pi}{6} - \cos\dfrac{4\pi}{3} + \tan\dfrac{2\pi}{3}$
16. $\sin\dfrac{2\pi}{3}\cos\dfrac{5\pi}{6} + \cos\dfrac{2\pi}{3}\sin\dfrac{5\pi}{6}$
17. $\cos\dfrac{3\pi}{4}\cos\dfrac{\pi}{4} - \sin\dfrac{3\pi}{4}\sin\dfrac{\pi}{4}$
18. $\sin\dfrac{11\pi}{6}\cos\dfrac{\pi}{3}\tan\dfrac{3\pi}{4}$
19. $\left(\cos\dfrac{11\pi}{6} + \sin\dfrac{\pi}{3}\right)\left(\tan\dfrac{\pi}{6} + \cot\dfrac{4\pi}{3}\right)$
20. $\left(\tan\dfrac{5\pi}{4} + \sin\dfrac{3\pi}{2}\right)\cos\dfrac{5\pi}{6}$

Find all the values for θ between 0 and 2π that satisfy each of the following equations.

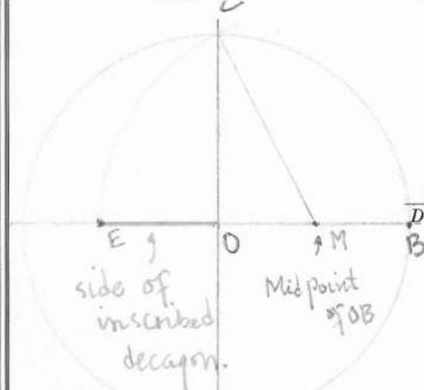
21. $\sin\theta = \frac{1}{2}$
22. $\cos\theta = -\frac{1}{2}$
23. $\tan\theta = 1/\sqrt{3}$
24. $\sin\theta = -\sqrt{3}/2$
25. $\tan\theta = -1$
26. $\cos\theta = -\sqrt{2}/2$
27. $\sin\theta = \sqrt{2}/2$
28. $\sec\theta = 2$
29. $\cot\theta = -\sqrt{3}$
30. $\csc\theta = 2/\sqrt{3}$
31. By drawing a figure for each of the following values of θ in the unit circle, make a table giving the six circular functions of each: 0, $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$, $2\pi/3$, $3\pi/4$, $5\pi/6$, π, $7\pi/6$, $5\pi/4$, $4\pi/3$, $3\pi/2$, $5\pi/3$, $7\pi/4$, $11\pi/6$, 2π.
32. Using Eq. (6–18), find the length of the chord in a unit circle if the corresponding θ is (a) $\pi/6$, (b) $\pi/3$, (c) $2\pi/3$.

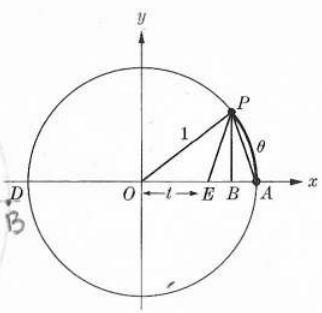
6–5 Exact values of the circular functions for $\theta = \pi/5$. There is one other special value which is of interest. Although a slightly more complicated construction is required for $\pi/5$, we are able to find exact values for the circular functions of this number, and thus in all have exact values for the functions of π, $\pi/2$, $\pi/3$, $\pi/4$, $\pi/5$, and $\pi/6$.

Consider the unit circle with center O and radius OA. (See Fig. 6–14.) Locate the point E on the radius OA so that

$$\frac{\text{Length } OA}{\text{Length } OE} = \frac{\text{Length } OE^*}{\text{Length } EA}.$$

* This location is possible with the use of ruler and compass, and is essentially the same as that used for constructing a regular pentagon.





MC = ME, where M is midpoint of OB Then OE is the side of inscribed decagon.

FIGURE 6–14

By letting t denote the length of OE, we have the relationship

$$\frac{1}{t} = \frac{t}{1-t}. \tag{6–19}^*$$

Solving this quadratic equation† for the positive value of t, we find $t = (\sqrt{5}-1)/2$.

Locate P so that the length of the chord AP is equal to t. Then the triangles OPA and AEP are similar, and thus triangle AEP is isosceles, so that the length of PE is equal to t, and triangle OEP is also isosceles. From this it follows directly that $\angle OAP$ is twice $\angle POA$. (See Problems 3, 4, and 5.) Since $\angle OAP$ is measured by $\frac{1}{2}\widehat{PD}$,‡ $\angle POA$ is measured by $\frac{1}{4}\widehat{PD}$. Also $\angle POA$ is measured by $\widehat{PA}$. Thus, the length of $\widehat{PD}$, $|\widehat{PD}| = 4|\widehat{AP}|$. But $|\widehat{PD}| + |\widehat{AP}| = \pi$. As a direct result, $|\widehat{AP}| = \pi/5$.

* The ratio defined by Eq. (6–19) has been considered since the time of the Greek mathematicians, and is called the *golden section*. Much has been written on this subject that would be of interest to the student. For example:

1. H. V. Barravalle, "The Geometry of the Pentagon and the Golden Section," *Mathematics Teacher*, Jan. 1948.
2. W. W. Rouse Ball, *Mathematical Recreations and Essays*, rev. by H. S. M. Coxeter, 11th Ed. London: Macmillan and Company, Ltd., 1940.
3. R. Courant and H. Robbins, *What Is Mathematics?*. New York: Oxford University Press, 1941.
4. Jay Hambridge, *The Elements of Dynamic Symmetry*. New York: Brentano's, 1926.
5. Tobias Dantzig, *Bequest of the Greeks*, 1955.

† The solution of quadratic equations is discussed in Article 7–4. (See Problem 3, second list.)

‡ Use is made of the theorem in plane geometry which states that an angle inscribed in a circle is measured by one-half the intercepted arc.

By considering PB perpendicular to OA at B, we easily find the exact lengths of OB and PB, that is, the x- and y-coordinates of $P = P(\pi/5)$, or the cosine and sine of $\pi/5$. The length of EA is

$$1 - t = 1 - \frac{\sqrt{5}-1}{2} = \frac{3-\sqrt{5}}{2};$$

thus the length of EB is $(3-\sqrt{5})/4$. Therefore the length of OB is

$$OB = \frac{\sqrt{5}-1}{2} + \frac{3-\sqrt{5}}{4} = \frac{\sqrt{5}+1}{4}.$$

Also, in the right triangle OPB,

$$(\text{length of } PB)^2 = 1 - \left(\frac{\sqrt{5}+1}{4}\right)^2 = \frac{5-\sqrt{5}}{8}.$$

Thus

$$\sin\frac{\pi}{5} = \sqrt{\frac{5-\sqrt{5}}{8}}, \qquad \cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}. \tag{6–20}$$

Should the numerical values of the other functions be desired, they can now be found.

PROBLEMS

Referring to Fig. 6–14, prove the following in detail.

1. Triangle AEP is isosceles.
2. Triangle OEP is isosceles.
3. Angles AOP, OPE, and EPA are equal.
4. $\angle OPA = \angle OAP$.
5. $\angle PAO =$ twice $\angle POA$.
6. Using Eq. (6–18), find the length of the chord in a unit circle if the corresponding θ is $\pi/5$. How does this value compare with the value of t?

6–6 The fundamental circular function identities. In the last few articles we have considered some of the useful relations between the different circular functions. There are eight of these, known as *the fundamental circular function identities*, and from these, other simple identities may be proved. They consist of the three reciprocal relations, Eqs. (6–11), (6–12), (6–13), the tangent and cotangent relations, Eqs. (6–4) and (6–14), and the Pythagorean relations, Eqs. (6–15), (6–16), (6–17).

With the use of these relations, all the circular functions may be expressed in terms of any one circular function. Consider the following example.