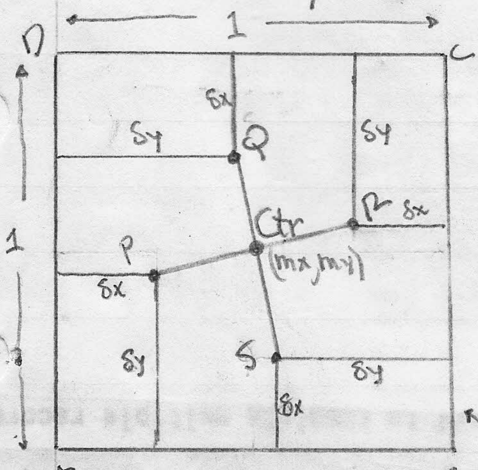
1977/1 Equilateral triangles ABK, BCL, CDM, DAN are constructed inside the square ABCD. Prove that the midpoints of the 4 segments KL, LM, MN, NK and the midpoints of the 8 segments AK, BK, BL, CL, CM, DM, DN, AN are the 12 vertices of a regular dodecagon.

Proof: Let P be the intersection of AK and BL. ΔAPB is a $30°-60°-90°$ Δ, so $AP = \frac{1}{2}\cdot AB = \frac{1}{2}\cdot AK$. That is, P is the midpoint AK. Similarly for:

See diagram next page.

Q = intersection of DN and AK
R = " " CM and DN
S = " " BL and CM

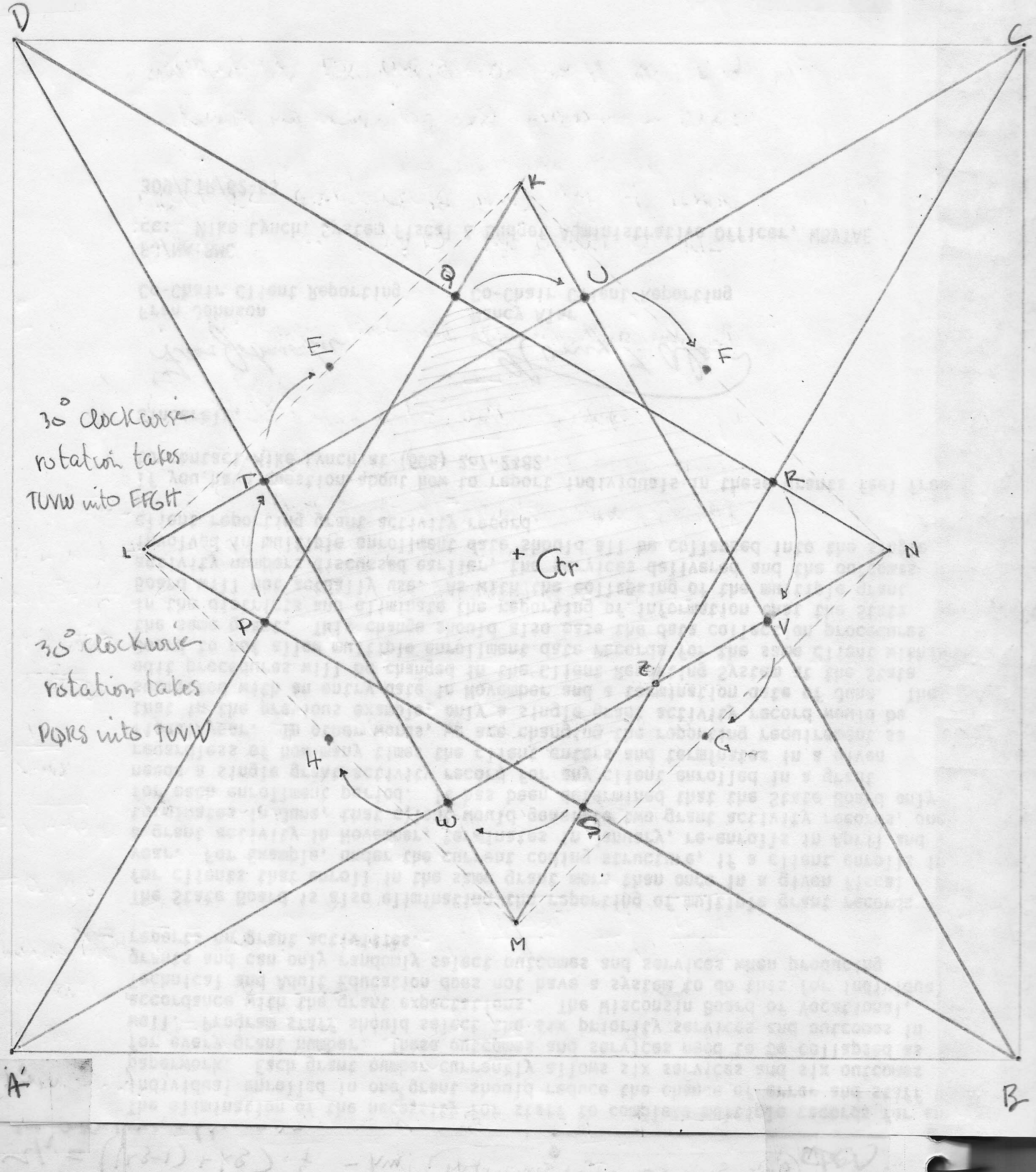
In fact PQRS is a square, since $BL \parallel DN$, $AK \parallel CM$ (The pairs are 30° off the horizontal and vertical respectively), and all the angles of PQRS are 90° (eg., in Δ QLC, $\angle L = 60°$, $\angle C = 30°$, so $\angle Q = 90°$ — similarly for the other angles of PQRS).



(δx's, δy's same by symmetry.)

Furthermore PQRS has the same center as the original square ABCD, Ctr. For wlog, let ABCD have side 1. Let (mx, my) = midpt of $\overline{PR}$, a diagonal of PQRS:

$$mx = \frac{1}{2}(\delta x + (1-\delta x)) = 1/2$$
$$my = \frac{1}{2}(\delta y + (1-\delta y)) = 1/2$$


D
C
K
Q
U
E
F
30° clockwise rotation takes TUVW into EFGH
T
R
L
N
Ctr
30° clockwise rotation takes PQRS into TUVW
P
V
Z
G
H
W
S
M
A
B

P, Q, R, S are 4 of the midpoints in the problem definition. Label the 4 other midpoints of the legs of the equilateral Δ's T, U, V, W — T is the intersection of DM and CL, with U, V, and W marked by proceeding clockwise.

The situation is identical to above, so TUVW is also a square centered at Ctr, and of the same size as PQRS. These 2 squares are offset by a rotation of 30° around their common center Ctr — rotating PQRS by 30° clockwise makes it coincide with TUVW. To see this, note that PS is at 30° wrt the horizontal, while TW is 60°.

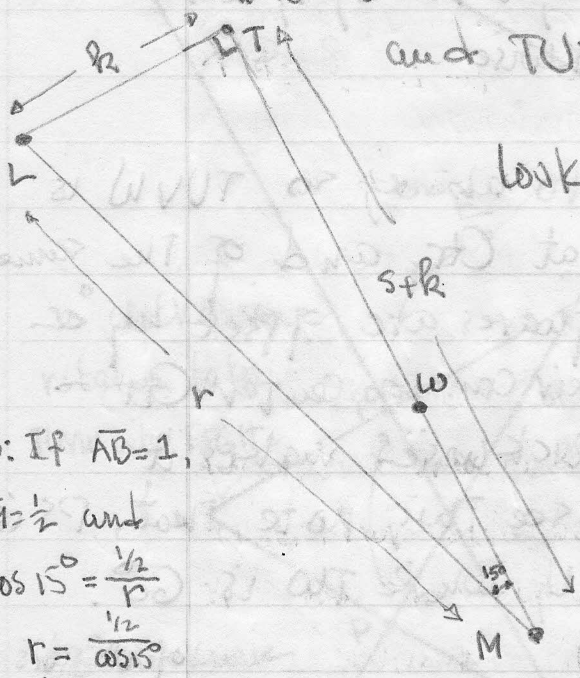
Now consider the quadrilateral LKNM. Each of its sides is the base of a Δ with sides from the equilateral Δs and top angle = 30°: eg, in Δ LBK, LB and BK are from the equilateral Δs and ∠B = 30°.

Let Z = intersection of VW and RS. Looking at quadrilateral SZVB, ∠SBV = 30° and there are 2 90° angles, so ∠SZV = 150°. Now focusing on isosceles Δ MZN (isosceles by symmetry): must have ∠ZMN = 15°. Similarly ∠LMW = 15°, so ∠LMN = ∠LMW + ∠DMC + ∠ZMN
= 15° + 60° + 15° = 90°.

Also: LKNM has center Ctr, since it's diagonals are $\parallel$ to ABCD, and right in its middle, horizontally and vertically.

Similarly for the other angles in quadrilateral LKNM, which is $\therefore$ a square.

Next we calculate the size of LKNM wrt PQRS and TUVW, the latter 2 the same size.



Look at $15°-75°-90°$ Δ LMT and let

$r = \overline{LM}$ = side of square LKMN,

$k = \overline{LT}$

$s = \overline{WT}$ = side of square TUVW.

Have $\overline{MT} = s+k$, since $\overline{WM} = \overline{LT}$ by symmetry.

NB: If $\overline{AB}=1$, $\overline{TM}=\frac{1}{2}$ and $\cos 15° = \frac{1/2}{r}$

$\therefore r = \frac{1/2}{\cos 15°} = \frac{1/2}{\frac{1}{2}\sqrt{2+\sqrt{3}}} = \frac{1}{\sqrt{2+\sqrt{3}}} = \sqrt{2-\sqrt{3}} = 0.517638$

This is the diameter of the dodecagon.

$\cos 15° = \frac{s+k}{r}$, $\sin 15° = \frac{k}{r}$, so $\boxed{\cos 15° - \sin 15° = \frac{s}{r}}$

$\cos 15°$ and $\sin 15°$ can be calculated with the half-angle formulas:

$$\frac{s}{r} = \cos 15° - \sin 15° = \sqrt{\frac{1+\cos 30°}{2}} - \sqrt{\frac{1-\cos 30°}{2}}$$

$$= \sqrt{\frac{1+\sqrt{3}/2}{2}} - \sqrt{\frac{1-\sqrt{3}/2}{2}}$$

$$= \sqrt{\frac{2+\sqrt{3}}{4}} - \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{1}{2}\cdot\left\{\sqrt{2+\sqrt{3}} - \sqrt{2-\sqrt{3}}\right\}$$

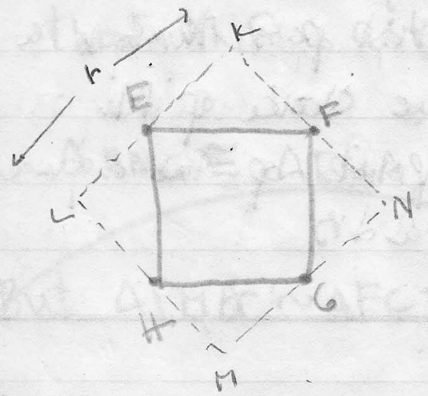
Let $x = \sqrt{2+\sqrt{3}} - \sqrt{2-\sqrt{3}}$.

Then $x^2 = (2 \cancel{+\sqrt{3}}) - 2 \cdot \sqrt{2+\sqrt{3}} \cdot \sqrt{2-\sqrt{3}} + (2 \cancel{-\sqrt{3}})$

$= 4 - 2 \cdot \sqrt{2^2 - \sqrt{3}^2} = 4 - 2 \cdot \sqrt{4-3} = 4 - 2 \cdot \sqrt{1}$

$= 4 - 2 = 2 \Rightarrow x = \sqrt{2}$

$\therefore \frac{s}{r} = \frac{1}{2} x = \frac{1}{2}\sqrt{2}$



But the square EFGH made by connecting the midpoints of square LKNM has the same ratio in size wrt LKNM:

$(\overline{EF})^2 = \left(\frac{r}{2}\right)^2 + \left(\frac{r}{2}\right)^2 = \frac{1}{2} r^2$

$\overline{EF}/r = \sqrt{1/2} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$.

$\therefore$ EFGH has the same size as PQRS and TUVW, and is also centered at Ctr (as LKNM is).

EFGH is rotated 30° cw wrt TUVW, because 30° rotates TW into EH, which is vertical.

The 3 interlaced squares, each rotated 30° cw wrt each other, form a regular dodecagon, centered at Ctr and with diameter = r = side of LKMN. $\boxed{\text{QED}}$