Nineteenth International Olympiad, 1977

1977/1 First solution. In Figure 52, O is the center of the square. Diagonals AC and BD are axes of symmetry, and so are the lines through M, K and through L, N. Rotation about O through any multiple of $90°$ also leaves the figure invariant. It is therefore enough to work in the portion of the plane bounded by rays OA and OL. Subsequent reflections in OA, then OM, and finally LN will generate the rest of the figure. We have denoted the midpoints of $AK, LM, AN, \cdots$ by $P_1, P_2, P_3 \cdots$ respectively.

Since $AK = AD$ and $\angle DAK = 90° - 60° = 30°$, we have $\angle ADK = \angle AKD = 75°$. Therefore the congruent isosceles triangles CDK,

BCN, ABM, DAL have base angles of $75° - 60° = 15°$.† It follows that congruent triangles AML, BNM, CKN and DLK are equilateral. Denote the length of their sides by s.

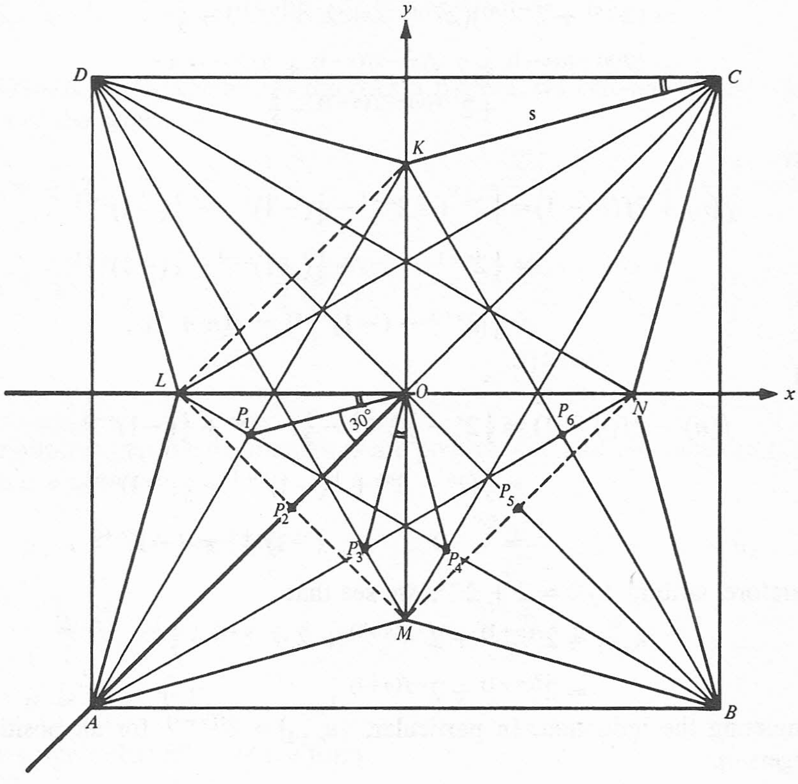

Figure 52

Segment OP_1 connecting the midpoints of sides AK and AC in $\triangle AKC$ is $\| KC$ and has length $\frac{1}{2} KC = \frac{1}{2} s$. By symmetry, $OL \| DC$, so $\angle LOP_1 = \angle DCK = 15°$. $\angle AOL = 45°$, so $\angle P_1OP_2 = 30°$. Segments AO and BO are perpendicular; BO bisects $\angle MBN$ and so is $\perp$ to MN. Hence $MN \| AO$ and $OP_2 = MP_5 = \frac{1}{2} s$; that is $OP_1 = OP_2$.

By reflection in line OA, we now have $OP_3 = OP_1$, $\angle P_2OP_3 = 30°$, $\angle P_3OM = 15°$. Next, reflect in line OM and add OP_4, OP_5 and OP_6 to the list of equal segments, and $\angle P_3OP_4 (= 2 \cdot 15°) = \angle P_4OP_5 = \angle P_5OP_6$ to the list of 30° angles.

†For more on this interesting configuration, see *Geometry Revisited* by H. S. M. Coxeter and S. L. Greitzer, NML vol. 19, 1967, p. 25.

Finally, reflection about LN produces the rest of the figure, and we have twelve points $P_1, P_2, \ldots P_{12}$ lying on a circle of radius $s/2$ and 30° apart on this circle. They are the twelve vertices of a regular dodecagon.

Second solution. Place the center O of the square at the origin of a coordinate plane so that the vertices of the square are the points

$$A = (-2, -2)\ ,\quad B = (2, -2)\ ,\quad C = (2, 2)\ ,\quad D = (-2, 2)\ .$$

Note that B, C, D may be obtained from A by successive rotations through 90° counterclockwise about the origin.

The altitude of equilateral $\triangle ABK$ with side length 4 has length $2\sqrt{3}$, so the coordinates of point K are $K = (0, 2\sqrt{3} - 2)$, see Figure 52. Successive rotations through 90° about O take point K into L, M, N; thus

$$K = (0, 2\sqrt{3} - 2)\ ,\quad L = (2 - 2\sqrt{3}, 0)\ ,$$

$$M = (0, 2 - 2\sqrt{3})\ ,\quad N = (2\sqrt{3} - 2, 0)\ .$$

We calculate the coordinates of the midpoints P_1, P_2, P_3 of segments AK, LM, AN, respectively, using the averages of the coordinates of the endpoints. We obtain

$$P_1 = (-1, \sqrt{3} - 2)\ ,\quad P_2 = (1 - \sqrt{3}, 1 - \sqrt{3})\ ,\quad P_3 = (\sqrt{3} - 2, -1)\ .$$

Now $\sqrt{3} - 2 < 0$ and $1 - \sqrt{3} < 0$, so P_1, P_2, P_3 are in the third quadrant. The other 9 midpoints can be obtained from P_1, P_2, P_3 by successive rotations through 90° about O. To prove that the twelve midpoints form a regular dodecagon, it suffices, thanks to the symmetry, to show that P_1, P_2, P_3 are equally distant from O and that sides P_1P_2, P_2P_3 and P_3P_4 of the dodecagon have the same length; here P_4 is the image of P_1 under rotation by 90° and hence has coordinates $P_4 = (2 - \sqrt{3}, -1)$. (The equality $P_1P_2 = P_2P_3$ follows from the symmetry with respect to the diagonal AC of the square.) We achieve this by using the distance formula:

$$OP_1^2 = 1 + (\sqrt{3} - 2)^2 = 8 - 4\sqrt{3}\ ,\quad OP_2^2 = 2(1 - \sqrt{3})^2 = 8 - 4\sqrt{3}\ ,$$

$$OP_3^2 = (\sqrt{3} - 2)^2 + 1 = 8 - 4\sqrt{3}\ ,$$

while

$$P_1P_2^2 = (\sqrt{3} - 2)^2 + (2\sqrt{3} - 3)^2 = 28 - 16\sqrt{3}\ ,$$

$$P_3P_4^2 = (2\sqrt{3} - 4)^2 = 28 - 16\sqrt{3}\ .$$

This shows that all midpoints have distance $2\sqrt{2 - \sqrt{3}}$ from O and all sides of the dodecagon have length $2\sqrt{7 - 4\sqrt{3}}$.

Remark. An alternative proof consists of subjecting the vector P_1 to three successive rotations of 30° for example by multiplying it by the rotation matrix

$$R = \begin{pmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

and then verifying that

$$RP_1 = P_2\,, \quad R^2P_1 = P_3 \quad \text{and} \quad R^3P_1 = P_4\,.$$