UW Talent Search Nov 2009 (Prob. Set II)

$a^2+b^2=c^2+5$:

a	b	c	a^2	b^2	c^2
0	3	2	0	9	4
1	2	0	1	4	0
4	5	6	16	25	36
6	15	16	36	225	256
7	10	12	49	100	144
8	29	30	64	841	900
9	18	20	81	324	400
10	47	48	100	2209	2304
11	28	30	121	784	900
12	69	70	144	4761	4900
13	40	42	169	1600	1764
14	95	96	196	9025	9216
15	54	56	225	2916	3136
17	70	72	289	4900	5184
19	88	90	361	7744	8100
20	37	42	400	1369	1764
21	108	110	441	11664	12100
25	26	36	625	676	1296
30	87	92	900	7569	8464
35	56	66	1225	3136	4356
40	67	78	1600	4489	6084
45	96	106	2025	9216	11236
48	51	70	2304	2601	4900
48	99	110	2304	9801	12100
59	68	90	3481	4624	8100

1. Show that the equation $a^2+b^2=c^2+5$ has infinitely many different positive integer solutions.

Proof: Computer search for solutions at left.

There are patterns:

a	b	c
7	10	12
9	18	20
11	28	30
13	40	42
⋮		

a	b	c
10	7	12
20	37	42
30	87	92
⋮		

a	b	c	k	2k	$(2k^2-3)$	$(2k^2-2)$
4	5	6	2	4	5	6
6	15	16	3	6	15	16
8	29	30	4	8	29	30
10	47	48	5	10	47	48
12	69	70	6	12	69	70
			⋮			

- over -

There is a hypothesis implicit in the third pattern (the others would do too):

let $a = 2k$, $b = 2k^2 - 3$, $k = 2, 3, 4, \dots$

$$a^2 + b^2 = 4k^2 + (2k^2 - 3)^2$$

$$= 4k^2 + (4k^4 - 12k^2 + 9)$$

$$= (4k^4 - 8k^2 + 4) + 5$$

$$= (2k^2 - 2)^2 + 5$$

So we have $a^2 + b^2 = c^2 + 5$ by

putting $c = 2k^2 - 2$ $(= b + 1)$.

QED.