Mathematics Magazine Problem 1456 (October 1994)

Show that the only sequence of numbers (α_i) that satisfies the conditions

$$(i) \qquad \alpha_i > 0 \quad \text{for all } i \geq 1, \quad \text{and}$$

$$(ii) \qquad \alpha_{i-1} = \frac{i\alpha_i + 1}{\alpha_i + i} \quad \text{for all } i > 0,$$

is the sequence $\alpha_i = 1$ for all i.

Solution. The two conditions imply that $i \geq 0$, while (ii) forces $\alpha_0 = 1$. Solving (ii) for α_i leads to

$$\alpha_i(\alpha_{i-1} - i) = 1 - i\alpha_{i-1}, \quad \text{and then} \tag{1}$$

$$\alpha_i = \frac{1 - i\alpha_{i-1}}{\alpha_{i-1} - i} \quad \text{if } \alpha_{i-1} \neq i. \tag{2}$$

If $\alpha_{i-1} = i$, then $1 - i\alpha_{i-1} = 0$ by (1), so $i^2 = 1$ and $i = 1$, showing that (2) holds for all $i > 1$.

Consider the family of functions

$$f_r(x) = \frac{1 - rx}{x - r} \quad \text{for fixed real } r > 0.$$

(2) implies that

$$\begin{aligned}
\alpha_2 &= \frac{1 - 2\alpha_1}{\alpha_1 - 2} = f_2(\alpha_1), \\
\alpha_3 &= \frac{1 - 3\alpha_2}{\alpha_2 - 3} = f_3(\alpha_2) = f_3(f_2(\alpha_1)), \\
\alpha_4 &= \frac{1 - 4\alpha_3}{\alpha_3 - 4} = f_4(\alpha_3) = f_4(f_3(f_2(\alpha_1))) \\
&\vdots
\end{aligned}$$

Direct calculation reveals that composing these functions results in a function of the same kind, with

$$(f_r \circ f_s)(x) = f_q(x), \quad \text{where } q = \frac{1 + rs}{r + s}.$$

Therefore there are constants r_i such that

$$F_i(x) \stackrel{\text{def}}{=} (f_i \circ f_{i-1} \circ \cdots \circ f_2)(x) = \frac{1 - r_i x}{x - r_i} \quad \text{for } i \geq 2.$$

Putting $r = i + 1$ and $s = r_i$ in the composition formula gives

$$r_{i+1} = \frac{1 + (i+1)r_i}{i + 1 + r_i} = \frac{1 + r_i + i + (ir_i - i)}{1 + r_i + i} = 1 + i \cdot \frac{r_i - 1}{1 + r_i + i}.$$

Since $r_2 = 2$, this proves that $r_i > 1$ for all $i \geq 2$, and therefore that

$$r_{i+1} - 1 = \frac{i}{1 + r_i + i} \cdot (r_i - 1) < \frac{i}{i+2} \cdot (r_i - 1).$$

Unfolding this expression leads to

$$r_{n+1} - 1 < (r_2 - 1) \cdot \prod_{i=2}^{n} \left(\frac{i}{i+2} \right) = \frac{3 \cdot 2}{(n+2) \cdot (n+1)}, \quad \text{for } n \geq 3,$$

the equality due to cancellation of all but two of the numerators and two of the denominators. It follows that $r_n \to 1$ as $n \to \infty$. The first few values of this sequence are

$$(r_i) = (2, \frac{7}{5}, \frac{11}{9}, \frac{8}{7}, \frac{11}{10}, \frac{29}{27}, \ldots), \quad \text{for } i \geq 2.$$

Now fix $x = \alpha_1 > 0$. Then by the original condition (i),

$$\alpha_i = F_i(x) = \frac{1 - r_i x}{x - r_i} > 0 \quad \text{for } i \geq 2.$$

This forces

$$\frac{1}{r_i} < x < r_i \quad \text{for } i \geq 2.$$

It follows that $x = \alpha_1 = 1$ and also that $\alpha_i = F_i(1) = 1$ for all $i > 1$.

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