Kepler's Nested Platonic Solids



Kepler's scheme in the *Mysterium Cosmographicum* nests the five Platonic solids in the orbits of the then (1596) six known planets. The nesting is tight, meaning that the innner orbit is tangent to the face of its circumscribing solid, while the outer orbit runs through the solid's vertices. The hyper-symmetry of each of these solids implies that its inscribed sphere touches all its faces and similarly the circumscribed sphere goes through all its vertices. Kepler had a cube between Jupiter and Saturn, the two outermost planets. Consider a cube of side one; the object is to find the ratio of the inscribed and circumscribed spheres — they are supposed to be the orbits of Jupiter an Saturn respectively. Note that the mean distance of Jupiter from the sun is 5.20 au (astronomical units, 1 au being the mean distance of the earth from the sun). Saturn's orbit averages 9.54 au, so their ratio is 5.20/9.54 = 0.545.



Segment AC in the figure above is the diagonal of a face, so obviously has length $\sqrt{2}$. Applying the Pythagorean Theorem to right triangle $\triangle CAC'$ inside the cube leads to $AC'^2 = AC^2 + CC'^2 = \sqrt{2}^2 + 1^2 = 2 + 1 = 3$, so

 $AC' = \sqrt{3}$. But AC', the diagonal connecting opposite vertices of the cube, is obviously twice the radius of the cube's circumscribing sphere, while the inscribed sphere has length 1/2. Therefore:

$$\frac{\text{radius of inscribed sphere}}{\text{radius of circumscribed sphere}} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = 0.577$$

This compares reasonably well to the ratio of the orbits of Jupiter and Saturn, given above as 0.545:

$$\operatorname{error}_{js} = \frac{0.577 - 0.545}{0.545} = 5.87\%$$
 (over).

Next is the tetrahedron, which Kepler put between the orbits of Mars and Jupiter. It's convenient to draw a tetrahedron inside a cube of side 1 as pictured here — note the six sides of the tetrahedron are all diagonals of the six faces of the cube. It's clear the green figure inside the cube is a regular tetrahedron, because all the faces are congruent equilateral triangles and the four solid angles are congruent as well.



The center of the tetrahedron C is the same as the center of the cube. The objective is to find the perpendicular distance from C to a face of the tetrahedron *and* the distance from C to a vertex of the tetrahedron, and to compare the two. Let B be the center of one of the tetrahedron's faces and A one of that face's vertices, as pictured.

Some plane geometry will show how distant the center of an equilateral triangle is from its side and vertex. We want to use this to analyse the equilateral triangle at the base of tetrahedron, the dark green one with B as center and A as one vertex.

In the diagram of an equilateral triangle below, the center is where the three altitudes meet and everything is perfectly symmetrical; in particular, the six smaller triangles making up the larger one, including the gray one, are all congruent and they are all 30% - 60% - 90% triangles. If the distance from a vertex to the center is 1, as pictured, then the perpendicular distance from the center to a side is 1/2 by analysing the little gray triangle (it's 30% - 60% - 90%). This shows that the center of an equilateral triangle is exactly twice as far from a vertex as it is from a side. So too the base of the little gray triangle is $\sqrt{3}/2$ and the side of the big triangle is $\sqrt{3}$. The upshot of all this is that in an equilateral triangle, the distance from the center to a vertex is the side divided by $\sqrt{3}$.



Returning to the tetrahedron in the cube, AC is half a main diagonal, which is $\sqrt{3}$, so $AC = \sqrt{3}/2$. B is the center of an equilateral triangle with side $\sqrt{2}$ (the length of the diagonals of the squares making up the cube), so AB is that side, $\sqrt{2}$, divided by $\sqrt{3}$. Applying the Pythagorean Theorem to $\triangle ABC$:

$$BC^{2} = AC^{2} - AB^{2} = \left(\frac{\sqrt{3}}{2}\right)^{2} - \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^{2} = \frac{3}{4} - \frac{2}{3} = \frac{1}{12}.$$
$$\therefore BC = \sqrt{\frac{1}{12}} = \frac{1}{2\sqrt{3}}.$$

Finally, what we really want:

$$\frac{\text{radius of inscribed sphere}}{\text{radius of circumscribed sphere}} = \frac{BC}{AC} = \frac{1/(2\sqrt{3})}{\sqrt{3}/2} = \frac{1}{2\sqrt{3}} \cdot \frac{2}{\sqrt{3}} = \frac{1}{3} = 0.333.$$

Mars is 1.52 au from the sun, Jupiter 5.20 au, with ratio 1.52/5.20 = 0.292, so:

$$\operatorname{error}_{mj} = \frac{0.333 - 0.292}{0.292} = 14.0\%$$
 (over).

Next Kepler puts a dodecahedron between Earth and Mars. Let r_i be the radius of the inscribed sphere of the dodecahedron, that is, the perpendicular distance from a face to the center; and r_c the radius of the circumscribed sphere, that is, the distance from a vertex to the center. The object is to calculate r_i/r_c . Cut out a pyramid from the dodecahedron as shown below, with base one of the pentagonal faces, height r_i , and slant height r_c . The pyramid is perfectly symmetrical, so the pyramid's top vertex lies directly above the center C of the pentagon; that is, a perpendicular line erected from the plane of the pentagon right from its center goes through the top vertex (actually the center of the dodecahedron).



Apply the Pythagorean Theorem to the indicated triangle inside the pyramid:

$$r_i^2 + k^2 = r_c^2$$

$$\therefore \left(\frac{r_i}{r_c}\right)^2 + \left(\frac{k}{r_c}\right)^2 = 1.$$
 (1)

The task now is to calculate each of k and r_c as a constant multiple of the dodecahedron's side s, which will lead to a numerical value for k/r_c , then in turn to a simple calculation for r_i/r_c , the final object.



It's clear from this figure of a regular pentagon that:

$$\sin 36^\circ = \frac{s/2}{k}$$
$$\therefore k = \frac{s}{2\sin 36^\circ}.$$

In order to calculate r_c , embed a cube in a dodecahedron as shown here:



The image on the left is from Kepler's *Harmonices Mundi* of 1619 (p 181). It shows how a regular dodecahedron can be thought of as a cube with little tents built over each face, each of the six. This is the construction Euclid used to construct the dodecahedron in the last proposition in *The Elements*, Book XIII, 17. Note that each edge of the cube is a diagonal of one of the pentagons making up the dodecahedron. So:

 $2r_c =$ diagonal of the inscribed cube = $\sqrt{3} \cdot ($ diagonal of a pentagon making up the dodecahedron)= $\sqrt{3} \cdot \varphi \cdot ($ side of a pentagon making up the dodecahedron)= $\sqrt{3}\varphi s.$

The second line follows because the diagonal of a pentagon is the same as an edge of the cube; also, because the main diagonal of a cube is $\sqrt{3}$ times its side, as came up earlier. The third line follows from the fact that the diagonal of a regular pentagon is φ times its side, where $\varphi = (\sqrt{5} + 1)/2$ is the golden ratio. So:

$$\frac{k}{r_c} = \frac{s/(2\sin 36^\circ)}{\sqrt{3\varphi s/2}} = 0.60706.$$

Plugging this value into (1) leads to:

$$\left(\frac{r_i}{r_c}\right)^2 + 0.60706^2 = 1,$$

so
$$\left(\frac{r_i}{r_c}\right)^2 = 1 - 0.60706^2 = 1 - 0.36852 = 0.63148$$

and
$$\frac{r_i}{r_c} = \sqrt{0.63148} = 0.79466 \sim 0.795.$$

Earth of course is 1 au from the sun, Mars 1.52 au, so the ratio is 1/1.52 = 0.658. The error is:

$$\operatorname{error}_{em} = \frac{0.795 - 0.658}{0.658} = 20.8\%$$
 (over).

Kepler puts the icosahedron between Earth and Venus and the octahedron between Venus and Mercury. To calculate their ratios of inscribed to circumscribed spheres, I'll appeal to a theorem that the ratio is the same for dual polyhedra (see *Regular Polytopes, 3rd ed*, 1973, by H. M. S. Coxeter, pp. 16-17). The same page of Kepler with the image above shows that the dodecaheddron and icosahedron are duals, as are the cube and the octahedron (*conjugia* in Latin, for union or marriage)

Look at the figure on the left below showing an octahedron inside a cube. Each vertex of the octahedron touches the center of one of the cube's faces, and they match up perfectly with each of the octahedron's six vertices touching one of the cube's six faces. Vice versa, a cube can be inscribed inside an octahedron, with each of the cube's *eight* vertices touching the center of one of the octahedron's faces (there are eight of them too). We say cube and the

octahedron are *dual polyhedra* and the ratios of the radii of their inscribed and circumscribed sheres are equal, namely $1/\sqrt{3}$. The figure on the right shows an icosahedron (twelve vertices) inside a dodecahedron (twelve faces). They too are dual and the ratios of their spheres are the same.



The accuracy of Kepler's scheme can now be evaluated, where the "Kepler Ratio" of a polyhedron is the ratio of the radii of its inscribed and circumscribed spheres:

Adjacent Planets	Ratio of Orbits	Polyhedron	Kepler Ratio	Error
Jupiter – Saturn	5.20 / 9.54 = 0.545	Cube	0.577	5.87%
Mars – Jupiter	1.52 / 5.20 = 0.292	Tetrahedron	0.333	14.0%
Earth – Mars	1.00 / 1.52 = 0.658	Dodecahedron	0.795	20.8%
Venus – Earth	0.723 / 1.00 = 0.723	Icosahedron	0.795	9.96%
Mercury – Venus	0.387 / 0.723 = 0.535	Octahedron	0.577	7.85%

There are four possible Kepler configurations considering that the cube and octahedron can be exchanged, and similarly for the dodecahedron and icosahedron. Other than that, Kepler's configuration is plainly the optimal fit.

Kepler proposed this scheme as a young man in 1596 in the *Mysterium Cosmographicum*. The two images above are from the *Harmonices Mundi* of 1619, a more mature work coming ten years after his discovery that planetary orbits follow elliptical paths. All the same, much of *Harmonices Mundi* is taken up with polyhedra, which continued to fascinate him as they have many others down through the ages. Kepler is recognized to this day as a pioneer in studying polyhedra, some of them being named after him (the Kepler–Poinsot star polyhedra).

Kepler's regular polyhedon model became obsolete with his own discovery of elliptical orbits short years later. The planetary distances used above are the modern figures for their mean distances from the sun (the average of the perihelion, or closest distance, and aphelion, or furthest distance). Kepler explores various simple mathematical and musical relationships satisfied by planetary distances in Book 5 of the *Harmonices Mundi*; it was his abiding conviction that the planets were governed by simple mathematical laws.

He includes figures for perihelion and aphelion in the *Harmonices Mundi* (p. 195) and they are quite accurate, presumably due to Tycho Brahe. For the earth, he gives the ratio of aphelion to perihelion as 1018/982, 1000 obviously being the normalized mean distance (1000 = 1 au). 1018/982 = 1.0367 compares well with the modern figure of 1.0339 — it is off by about 0.26%. His values for Jupiter's aphelion and perihelion are very accurate, their ratio differing from the modern value by less than 0.13%. He is characteristically interested in comparing values with small proportions, noting that $1018/982 \sim 25/24$, for example.