$\sin(\frac{1}{2}\pi - \lambda)x$. (These follow since, e.g., $\cos(\mu n + \lambda)x = \cos(\pi/2 + \lambda)x = \cos \pi a \cdot \cos(\pi/2 - \lambda)x + \sin \pi a \cdot \sin(\pi/2 - \lambda)x$ for the π values of x we are considering.) But Gauss notes that by calculation $k \sin \pi a/2 = l \cos \pi a/2$, and hence

$$\frac{1}{2}(k + k \cos \pi a + l \sin \pi a) = k,$$

$$\frac{1}{2}(l + k \sin \pi a - l \cos \pi a) = l.$$

Thus in X'' he finds that γ^λ leads to the term $k \cos(\pi/2 - \lambda)x + l \sin(\pi/2 - \lambda)x$, and that δ^λ leads to $l \cos(\pi/2 - \lambda)x - k \sin(\pi/2 - \lambda)x$.

To illustrate these ideas Gauss took data on the position of the asteroid Pallas from Baron von Zach's tables. The data tabulated below from Gauss's paper give the declination X of Pallas as a function of the right ascension x. (The abbreviations *Austr.* and *Bor.* stand for the adjectives *australis*, southern, and *borealis*, northern.) We notice there are twelve values of x which span the period of 360°. Thus $\pi = 12$.

x	X	
0°	6°48′ Bor....	= + 408′
30	1 29	+ 89
60	1 6 Austr.....	− 66
90	0 10 Bor	+ 10
120	5 38	+ 338
150	13 27	+ 807
180	20 38	+1238
210	25 11	+1511
240	26 23	+1583
270	24 22	+1462
300	19 43	+1183
330	13 24	+ 804

Now, to form the finite Fourier series for this function, Gauss broke up the values of x into three groups of four each as shown below.

$a =$ 0°	$A = +$ 408	$a' =$ 30°	$A' = +$ 89	$a'' =$ 60°	$A'' = -$ 66
$b =$ 90°	$B = +$ 10	$b' =$ 120°	$B' = +$ 338	$b'' =$ 150°	$B'' = +$ 807
$c =$ 180°	$C = +$1238	$c' =$ 210°	$C' = +$1511	$c'' =$ 240°	$C'' = +$1583
$d =$ 270°	$D = +$1462	$d' =$ 300°	$D' = +$1183	$d'' =$ 330°	$D'' = +$ 804

and then formed the functions

$$X' = \gamma + \gamma' \cos x + \gamma'' \cos 2x + \delta' \sin x + \delta'' \sin 2x \tag{4.65}$$

for each group. ($\mu = 4$, $\nu = 3$; this is case (II) above where $n = 1$, $m = 2$.)

Thus he found three Fourier expansions:

Pro periodo	ubi $y = 4x$	γ	γ'	δ'	γ''	δ''
prima	0°	+779.5	−415.0	−726.0	+43.5	0
secunda	120°	+780.2	−404.5	−721.4	+ 9.9	+17.1
tertia	240°	+782.0	−413.5	−713.3	+11.7	−20.3

— one for each group.

Gauss now viewed each coefficient $\gamma, \gamma', \delta', \gamma'', \delta''$ as a periodic function of the variable $y(= 4x)$ and formed its Fourier series. This gave him the expansion for γ,

$$\frac{1}{3}(779.5 + 780.2 + 782.0)$$

$$+ \frac{2}{3}(779.5 + 780.2 \cos 120° + 782.0 \cos 240°) \cos 4x$$

$$+ \frac{2}{3}(780.2 \sin 120° + 782.0 \sin 240°) \sin 4x,$$

and after simplification he found the expansions

$$\begin{aligned}
\gamma &= 780.6 - 1.1 \cos 4x - 1.0 \sin 4x, \\
\gamma' &= -411.0 - 4.0 \cos 4x + 5.2 \sin 4x, \\
\delta' &= -720.2 - 5.8 \cos 4x - 4.7 \sin 4x, \\
\gamma'' &= +21.7 + 21.8 \cos 4x - 1.1 \sin 4x, \\
\delta'' &= -1.1 + 1.1 \cos 4x + 21.6 \sin 4x.
\end{aligned} \tag{4.66}$$

Now when these are substituted into the formula (4.65) above for X', and a little manipulation is performed, there results the expansion

$$\begin{aligned}
&780.6 - 411.0 \cos x - 720.2 \sin x + 43.4 \cos 2x - 2.2 \sin 2x \\
&\quad - 4.3 \cos 3x + 5.5 \sin 3x - 1.1 \cos 4x - 1.0 \sin 4x + 0.3 \cos 5x - 0.3 \sin 5x \\
&\quad + 0.1 \cos 6x.
\end{aligned}$$
[84]

Recall from our previous discussion that in the present case (II) Gauss showed that the $(\mu n + m)$th term in X'' was $\frac{1}{2}(k - l') \cos (\mu n + m)x + \frac{1}{2}(l + k') \sin (\mu n + m)x$, where γ^m, δ^m contained the terms $k \cos \mu nx + l \sin \mu nx$, $k' \cos \mu nx + l' \sin \mu nx$, respectively. Hence in this case, where $\mu = 4, \nu = 3, m = 2, n = 1$, we have $k = 21.8, l = -1.1, k' = 1.1, l' = 21.6$; and we see that the expression above reduces to $0.1 \cos 6x$, since $l + k' = 0$.

Next Gauss chooses $\mu = 3$, $\nu = 4$. Then we have his case (III), and $3n + m = 6 + m$, and $n = 2$. In this situation he divides up his 12 values into four groups of three each and writes $X' = \gamma + \gamma' \cos x + \delta' \sin x$. He finds now

Pro periodo	ubi $y = 3x$	γ	γ'	δ'
prima	0°	+776.3	−368.3	−718.8
secunda	90°	+786.0	−414.5	−676.0
tertia	180°	+785.0	−453.0	−721.1
quarta	270°	+775.0	−408.2	−765.0,

[84] Gauss III *TI*, p. 310. (The *signum* of 411.0 is erroneously given there as +.)