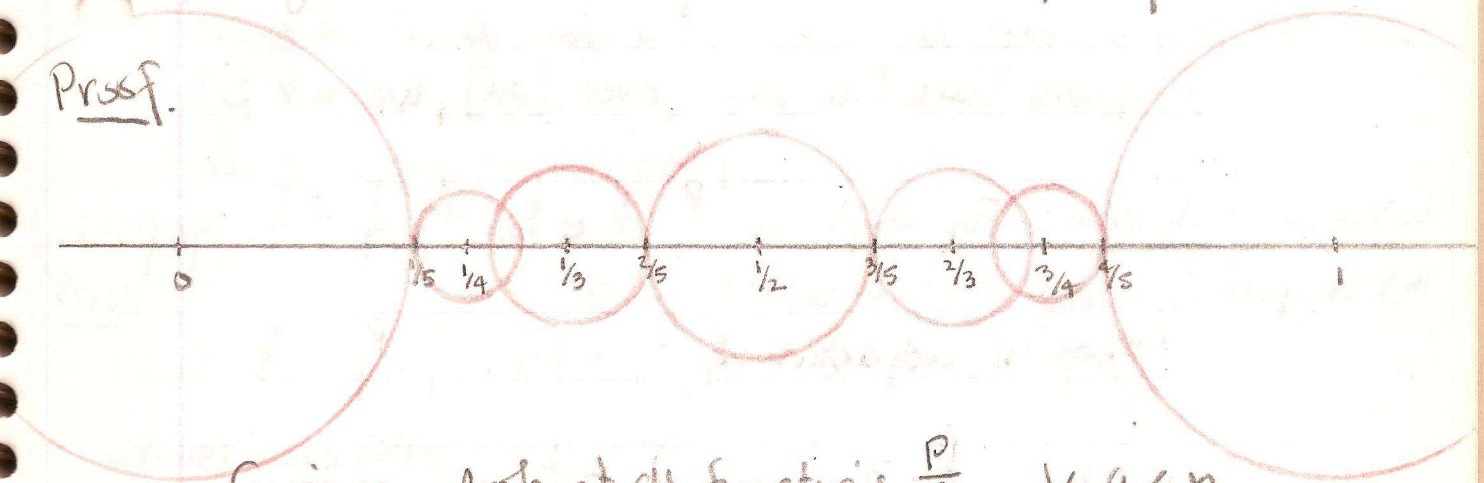
1928/1 Prove that, among the positive numbers $\{a, 2a, \dots (n-1)a\}$, there is one that differs from an integer by at most $1/n$.

Proof.



Given n, look at all fractions $\frac{p}{q}$, $1 \le q < n$
$0 \le p \le q$, $(p,q)=1$

These are the Farey fractions F_{n-1} of order $n-1$.
Eg., for $n=5$, $F_4 = \{0, 1/4, 1/3, 1/2, 2/3, 3/4, 1\}$.

For each $p/q \in F_{n-1}$, put closed ball (interval) centered at p/q with radius $1/qn$.

The illustration shows $n=5$ with balls:

center	radius
0, 1	1/5
1/2	1/10
1/3, 2/3	1/15
1/4, 3/4	1/20

The theorem follows if we could prove that these balls cover $[0,1]$. For wlog $0 \le a \le 1$ and:

$$|a - p/q| \le \frac{1}{qn} \Rightarrow |qa - p| \le \frac{1}{n}.$$

ie, if a is in the ball around p/q, then qa is within $1/n$ of integer p (where $q<n$).

To prove that the balls cover $[0,1]$, we use:

LeVeque Thm 8-14, p155

$\boxed{\underline{\text{Theorem}}: \text{ For adjacent } \frac{p}{q}, \frac{r}{s} \in F_{n-1},\ \frac{p}{q} < \frac{r}{s},\ qr-ps=1.}$

Consider adjacent $\frac{p}{q}, \frac{r}{s} \in F_{n-1}$, $\frac{p}{q} < \frac{r}{s}$.

Want: $\frac{p}{q} + \frac{1}{qn} \geq \frac{r}{s} - \frac{1}{sn}$ { This means that the balls overlap, at least in 1 point.

Mqsn: $psn + s \geq rqn - q$

$$s+q \geq rqn - psn = n\cdot(\underbrace{rq-ps}_{1}) = n.$$

But the latter inequality is true. To see this, note $\frac{p}{q} < \frac{p+r}{q+s} < \frac{r}{s}$.

$$\left[\begin{array}{l} p\cdot(q+s) < (p+r)q \Leftrightarrow ps < rq \\ (p+r)s < r(q+s) \Leftrightarrow ps < rq \end{array}\right]$$

Since $\frac{p}{q}, \frac{r}{s}$ are <u>adjacent</u> elements of F_{n-1},

There is <u>not</u> another element of F_{n-1} between them.

This implies $q+s \geq n$. QED.