1928/1. Prove that, among the positive numbers

$$a,\ 2a,\ \cdots,\ (n-1)a,$$

there is one that differs from an integer by at most $1/n$.

Solution. The problem states a special case of the theorem of Dirichlet–Kronecker† which may be proved by the box principle (cf.

† L. Dirichlet (1805–1859) was professor first in Berlin and then (as Gauss' successor) in Göttingen. His best-known theorem is:

If a and d are relatively prime, then the infinite arithmetic progression

$$a,\ a+d,\ a+2d,\ \cdots$$

contains infinitely many primes.

L. Kronecker (1823–1891) was professor in Berlin.

1906/3, Note). Roughly, it states that, for large n, at least one of these numbers is "almost" an integer. The theorem states exactly how to interpret this "almost".

We use the device shown in Figure 91 for the case $n = 12$; that is, we draw a circle, use its circumference as the unit of length, and mark off the $n - 1$ lengths $a, 2a, \cdots, (n-1)a$ along the circumference, beginning at some point O. (A circle of circumference 1 is convenient because we are interested only in the fractional portions of our numbers. For example, $e = 2.717\cdots$ and $.717\cdots$ are equivalent in this problem.)

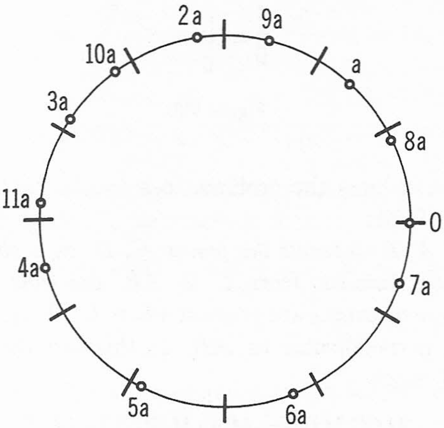

Figure 91

Next, we divide the circle into n equal arcs of length $1/n$, starting at the point O; we consider a division point as belonging only to the arc of which it is the endpoint. In this way, every point on the circle belongs to exactly one arc.

Our object is to show that one of the arcs adjacent to point O necessarily contains one of our $n - 1$ marked points. If this were not so, then all $n - 1$ marked points would belong to the $n - 2$ remaining arcs and hence, by the box principle (see 1906/3, Note), one of these $n - 2$ arcs must contain two of the marked points. In the figure the points $3a$ and $10a$ are on the same arc, and so their difference is less than $1/n$. But then the distance between the division point at O and the marked point

$$10a - 3a = 7a$$

is also less than $1/n = 1/12$ which shows, contrary to our assumption, that one of the arcs adjacent to O must contain one of the $n - 1$ points. In this case, $7a$ differs from an integer by at most $1/12$.

We used the illustrative example $n = 12$ and spoke of the numbers $11a$, $3a$, $10a$, and $7a$ instead of the corresponding numbers $(n-1)a$, ma, ka, and $(k-m)a$ that figure in the general case.