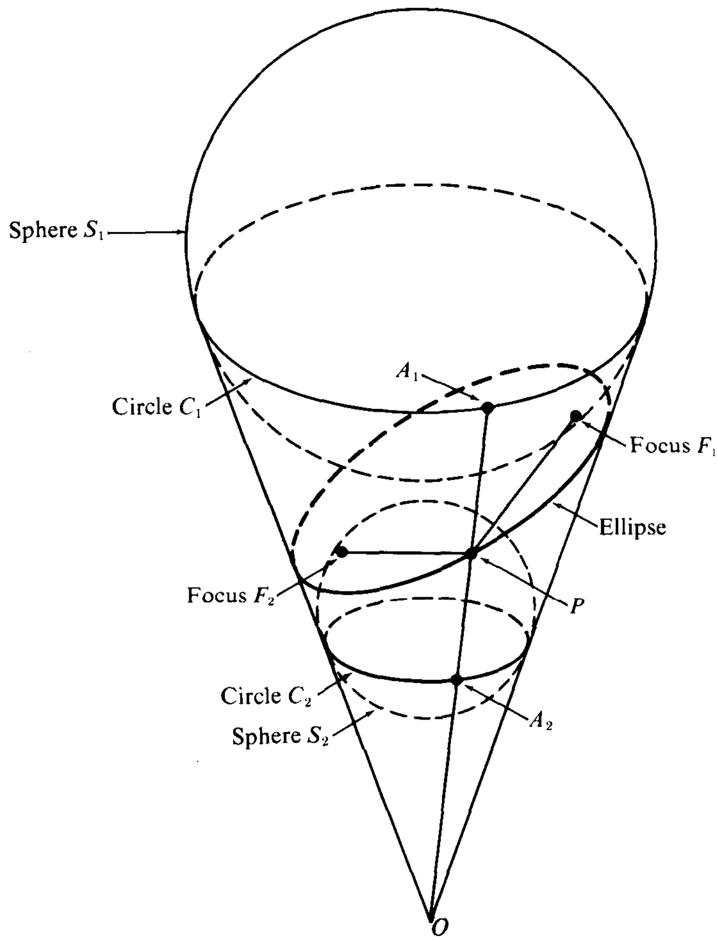

FIGURE 13.11 The ice-cream-cane proof.

Let P be an arbitrary point of the ellipse. The problem is to prove that $\|\overrightarrow{PF_1}\| + \|\overrightarrow{PF_2}\|$ is constant, that is, independent of the choice of P. For this purpose, draw that line on the cone from the vertex 0 to P and let A_1 and A, be its intersections with the circles C_1 and C_2, respectively. Then $\overrightarrow{PF_1}$ and $\overrightarrow{PA_1}$ are two tangents to S_1 from P, and hence $\|\overrightarrow{PF_1}\| = \|\overrightarrow{PA_1}\|$. Similarly $\|\overrightarrow{PF_2}\| = \|\overrightarrow{PA_2}\|$, and therefore we have

$$\|\overrightarrow{PF_1}\| + \|\overrightarrow{PF_2}\| = \|\overrightarrow{PA_1}\| + \|\overrightarrow{PA_2}\| .$$

But $\|\overrightarrow{PA_1}\| + \|\overrightarrow{PA_2}\| = \|\overrightarrow{A_1A_2}\|$, which is the distance between the parallel circles C_1 and C_2 measured along the surface of the cone. This proves that F_1 and F_2 can serve as foci of the ellipse, as asserted.

Modifications of this proof work also for the hyperbola and the parabola. In the case of the hyperbola, the proof employs one sphere in each portion of the cone. For the