MATH. MAG Dec 1998 #1562

Prove that: $\tan\left(\frac{1}{4}\tan^{-1}4\right) = 2\left(\cos\frac{6\pi}{17} + \cos\frac{10\pi}{17}\right)$.

See 'Galois Theory (2nd ed)' by Ian Stewart (1989) p 170ff

Let $\theta = 2\pi/17 = 0.3696 \equiv 21.1765°$.

and $\epsilon_k = e^{ki\theta} = \cos k\theta + i\sin k\theta$, $1 \le k \le 16$.

these are the 17^{th} roots of unity (other than 1)

and $\therefore$ satisfy : $\frac{t^{17}-1}{t-1} = t^{16} + t^{15} + \ldots + t + 1$.

Let $x_1 = \epsilon_1 + \epsilon_2 + \epsilon_4 + \epsilon_8 + \epsilon_{16} + \epsilon_{15} + \epsilon_{13} + \epsilon_9$

where each subscript is double the previous one (mod 17)

and similarly $x_2 = \epsilon_3 + \epsilon_6 + \epsilon_{12} + \epsilon_7 + \epsilon_{14} + \epsilon_{11} + \epsilon_5 + \epsilon_{10}$

Now $\epsilon_k + \epsilon_{17-k} = 2\cos k\theta$, so:

$x_1 = (\epsilon_1 + \epsilon_{16}) + (\epsilon_2 + \epsilon_{15}) + (\epsilon_4 + \epsilon_{13}) + (\epsilon_8 + \epsilon_9)$

$= 2(\cos\theta + \cos 2\theta + \cos 4\theta + \cos 8\theta)$

$x_2 = (\epsilon_3 + \epsilon_{14}) + (\epsilon_6 + \epsilon_{11}) + (\epsilon_{12} + \epsilon_5) + (\epsilon_7 + \epsilon_{10})$

$= 2(\cos 3\theta + \cos 5\theta + \cos 6\theta + \cos 7\theta)$

Because the ϵ_k are the powers of ϵ_1,

and $1 + \sum_{k=1}^{16} \epsilon_1^k = 0$:

$$x_1 + x_2 = -1$$

and using $2\cos m\theta \cos n\theta = \cos(m+n)\theta + \cos(m-n)\theta$

$$x_1 x_2 = 4(\cos\theta + \cos 2\theta + \cos 4\theta + \cos 8\theta)\cdot$$
$$(\cos 3\theta + \cos 5\theta + \cos 6\theta + \cos 7\theta) =$$

$$= 2(\cos 2\theta + \cos 4\theta + \cos 4\theta + \cos 6\theta +$$
$$\cos 5\theta + \cos 7\theta + \cos 6\theta + \cos 8\theta +$$
$$\cos\theta + \cos 5\theta + \cos 3\theta + \cos 7\theta +$$
$$\cos 4\theta + \cos 8\theta + \cos 5\theta + \cos 9\theta +$$
$$\cos\theta + \cos 7\theta + \cos\theta + \cos 9\theta +$$
$$\cos 2\theta + \cos 10\theta + \cos 3\theta + \cos 11\theta +$$
$$\cos 5\theta + \cos 11\theta + \cos 3\theta + \cos 13\theta +$$
$$\cos 2\theta + \cos 14\theta + \cos\theta + \cos 15\theta)$$

$\cos 2\theta = \cos 15\theta$
$\cos 3\theta = \cos 14\theta$
$\cos 4\theta = \cos 13\theta$
$\cos 6\theta = \cos 11\theta$
$\cos 7\theta = \cos 10\theta$
$\cos 8\theta = \cos 9\theta$

$$= 2(4\cos\theta + 4\cos 2\theta + 4\cos 3\theta + 4\cos 4\theta +$$
$$4\cos 5\theta + 4\cos 11\theta + 4\cos 7\theta + 4\cos 8\theta)$$

$$= 2(-2) = -4$$

since $\sum_{k=1}^{16} \cos k\theta = -1$, the real parts of $\sum_{k=1}^{16} \epsilon_1^k$ on the left.

$$\therefore \quad x_1 + x_2 = -1 \qquad x_1, x_2 = \frac{-1 \pm \sqrt{17}}{2}$$

$$x_1 \cdot x_2 = -4 \qquad = 1.5614, -2.5614$$

(Eq 1) $\therefore x_1, x_2$ roots of $t^2 + t - 4 = 0.$

Let $y_3 = \epsilon_3 + \epsilon_{12} + \epsilon_{14} + \epsilon_5$

and $y_4 = \epsilon_6 + \epsilon_7 + \epsilon_{11} + \epsilon_{10}$

so: $y_3 + y_4 = x_2$

$$y_3 = (\epsilon_3 + \epsilon_{14}) + (\epsilon_{12} + \epsilon_5)$$

$$= 2(\cos 3\theta + \cos 5\theta) \quad \leftarrow \text{positive}$$

$$y_4 = (\epsilon_6 + \epsilon_{11}) + (\epsilon_7 + \epsilon_{10})$$

$$= 2(\cos 6\theta + \cos 7\theta) \quad \leftarrow \text{negative}$$

$$\therefore y_3 y_4 = 4(\cos 3\theta + \cos 5\theta)(\cos 6\theta + \cos 7\theta)$$

$$= 2(\underline{\cos 3\theta} + \cos 9\theta + \underline{\cos 4\theta} + \cos 10\theta + \underline{\cos\theta} + \underline{\cos 11\theta} + \underline{\cos 2\theta} + \underline{\cos 12\theta})$$

$$= 2(\cos\theta + \cos 2\theta + \cos 3\theta + \cos 4\theta + \cos 5\theta + \cos 6\theta + \cos 7\theta + \cos 8\theta)$$

$$= 2(-1/2) = -1$$

(Eq 2)

$$\boxed{\begin{array}{ll} y_3 + y_4 = x_2 & \therefore y_3, y_4 \text{ roots of} \\ y_3 \cdot y_4 = -1 & t^2 - x_2 t - 1 = 0 \end{array}}$$

Let φ be smallest acute angle $\ni$ $\tan 4\varphi = 4$

$\therefore \cot 4\varphi = 1/4$ $\quad$ $4\varphi = \tan^{-1} 4$

$\varphi = \frac{1}{4}\tan^{-1} 4$

(Eq 1) and $t^2 + t - 4 = 0$ becomes:

$$t^2 + 4\cot 4\varphi - 4 = 0$$

whose roots are $t = 2\tan 2\varphi, -2\cot 2\varphi$

since $(2\tan 2\varphi)(-2\cot 2\varphi) = -4$

and $2\tan 2\varphi - 2\cot 2\varphi = 2\tan 2\varphi - \frac{2}{\tan 2\varphi}$

$$= 2\left\{\frac{\tan^2 2\varphi - 1}{\tan 2\varphi}\right\} = 2\{-2\cot 4\varphi\} = -4\cot 4\varphi$$

$\therefore x_1 = 2\tan 2\varphi$ and $x_2 = -2\cot 2\varphi$

(Eq 2) $\therefore$ $t^2 - x_2 t - 1 = 0$ becomes:

$$t^2 + 2\cot 2\varphi - 1 = 0$$

whose roots are $t = \tan\varphi$, $-\cot\varphi$

since $\tan\varphi \cdot (-\cot\varphi) = -1$

and $\tan\varphi - \cot\varphi = \dfrac{\tan^2\varphi - 1}{\tan\varphi} = -2\cot 2\varphi$
$= X_2$.

Equating the two expressions for Y_3, the positive root of (Eq 2):

$$Y_3 = 2(\cos 3\theta + \cos 5\theta)$$

$$Y_3 = \tan\varphi = \tan\left(\tfrac{1}{4}\tan^{-1} 4\right)$$

$$\boxed{\therefore \tan\left(\tfrac{1}{4}\tan^{-1} 4\right) = 2(\cos 3\theta + \cos 5\theta)}$$

QED.